FREE VIBRATION ANALYSIS OF HYBRID LAMINATED COMPOSITE PLATES WITH TRANSVERSE SHEAR

by

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INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MAY, 1988

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OF
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WITH
TRANSVERSE SHEAR

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

by

SHANKARA, C. A.

to the

DEPARTMENT OF AERONAUTICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MAY, 1988

to my beloved parents

CERTIFICATE

This is to certify that the work 'FREE VIBRATION ANALYSIS OF HYBRID LAMINATED COMPOSITE PLATES WITH TRANSVERSE SHEAR' has been carried out under my supervision and any part of this work has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENTS

I express my deep sense of gratitude to Professor N.G.R. Iyengar for his invaluable guidance and constant encouragement throughout the course of this work.

I sincerely thank Professor S. Durvasula for many fruitful discussions during crucial stages of this work.

I sincerely express my great indebtedness to my friends K. Kamal, K.G. Shastri, K.R. Upadhyaya, Gajbir Singh, Rajeevlochan, Aravind Galagali and Ramaprasad for several useful discussions.

I also thank my friends K.V. Shantha Ram, K.S.Srivatsa, K.S. Nagaraj, Vijay Kumar, Appu and many other friends for their direct and indirect help.

I owe to my brothers and sisters-in-law for their love and encouragement which helped me to complete this work.

Thanks are due to Mr. U.S. Mishra for neat typing of the manuscript.

-C.A. SHANKARA

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LIST OF SYMBOLS

NL	Number of plies in a laminate
a,b	Sides of the laminate
$t^{(k)}$	Thickness of k^{th} ply in the laminate
h	Thickness of the laminate
ρ	Average mass density of the material in the laminate
θ	Fibre orientation
u_o, v_o	Midplane displacements in the X and Y directions respectively
W_o	Transverse displacement
β_x, β_y	Rotations of the normals to the midplane in X and Y directions
$E_L^{(k)}, E_T^{(k)}$	Young's modulus of the material in the k^{th} lamina in longitudinal and transverse directions
$G_{LT}^{(k)}, G_{LZ}^{(k)}, G_{TZ}^{(k)}$	Shear moduli in respective directions
$\nu_{LT}^{(k)}, \nu_{TL}^{(k)}$	Major and minor Poisson's ratio of the material in the k^{th} lamina
$\{\sigma i\}$	Midplane stresses
$\{\varepsilon_i\}$	Midplane strains
A_{ij}	Inplane stiffness coefficients
B_{ij}, E_{ij}	Inplane/Flexural coupling coefficients
D_{ij}, F_{ij}, H_{ij}	Stiffness coefficients(flexural)
$[Q_{ij}]^k$	Reduced stiffness matrix of the k^{th} lamina

$[\bar{Q}_{ij}]^k$	Transformed reduced stiffness matrix of the kth lamina
$(N_i, Q_i, M_i, P_i, R_i)$	Stress resultants
(P, q_i, I_i)	Normal, Normal-rotatory, rotary inertia terms
U	Strain energy
T	Kinetic energy
ω_n	Natural frequency
K_n	Non-dimensional frequency parameter, $\omega_n a^2 \sqrt{\frac{\rho}{ETh^2}}$
m, n, r, s,	Integer indices in the deflection series
M	Maximum value of the indices m and r
N	Maximum value of the indices n and s
λ	Plate aspect ratio, a/b
ξ, η	Normalised values of x and y
L, T	Rectangular coordinates in the longitudinal and transverse directions
X, Y, Z,	Reference coordinates measured from one corner of the midplane of the plate
$\phi_i(\xi)$, $\psi_i(\eta)$	Admissible functions in ξ and η directions
II^{pq}_{mr}, JI^{pq}_{ns}	Integrals.

ABSTRACT

In the present work an attempt has been made to investigate the free vibration response of hybrid laminated composite plates made of Graphite/Epoxy and Kevlar/Epoxy laminae. Higher order shear deformation theory has been employed to take into account the transverse shear and rotatory inertia effects on the natural frequencies. Rayleigh-Ritz energy technique has been employed to solve vibration problem. Convergence studies have been carried out by considering number of terms in the displacements and rotations.

The validity of the present formulation has been checked by comparing the non-dimensional frequency parameter values with those obtained by other investigators for isotropic and regular composite laminates. Symmetric as well as antisymmetric angleply and crossply plates with simply supported edge conditions have been investigated. The effect of various parameters such as number of layers, fibre orientation, aspect ratio, length to thickness ratio etc. have been studied and the results are presented for regular as well as hybrid composite plates.

The Eigen-value problem has been solved using a standard subroutine available in DEC-1090 system. It has been found that the transverse shear effects cannot be neglected for plates having length to thickness ratio less than or equal to 20. The frequency parameter increases with length to thickness ratio thus proving that transverse shear effects can be neglected for thin plates without any appreciable loss in the accuracy of the results. The frequency parameter value increases with aspect ratio and number of layers. This attains an asymptotic value as the number of layers is increased. Frequency parameter has highest value for fibre orientations of $\pm 45^{o}$ since the stiffness of the plate is maximum for these fibre orientations. It has been found that hybrid laminates give intermediate properties as compared to those of single material laminates. It has been observed that hybrid laminates give values almost nearer to those of the Graphite/Epoxy laminates for aspect ratios up to 1. For aspect ratios upto 3, the frequency parameter reaches asymptotic value beyond length to thickness ratio of 30, while for higher aspect ratios, transverse shear effects are found to be predominant for length to thickness ratios higher than 30 also.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

Over the last two to three decades the use of composite materials consisting of either continuous or discontinuous fibres dispersed in a suitable metallic or nonmetallic matrix material has received wider attention. Due to the inherent tailoring of properties of these materials a number of unique design features can be utilised, including such potential aspects as reduction in weight, improved performance, increased service life, reduced system maintenance and active controlled configuration of structures. In aircrafts controlling the deflected shape of the wing during flight is of paramount importance.

Structural elements such as plates have found wide spread application in aeronautical, automobile, civil, marine, mechanical and other fields of engineering. They are subjected to various loading environment. It becomes imperative to have a thorough knowledge of the structural response, so as to enable one to achieve satisfactory design of these elements.

Laminated plates have been extensively used in the aerospace engineering industry. In laminated composite plate construction, high strength, low density fibres are dispersed at desired orientations in a matrix of resin.

Such a construction results in heterogeneous, anisotropy and the behaviour of these plates becomes highly complicated. To exploit the advantages of anisotropy, one should thoroughly investigate structural response due to transverse inplane and dynamic loads.

Extensive research has been carried out over the past two decades and enough information has been documented on structural behaviour of regular composite plates. It is well known that structural elements made of two or more different materials known as hybrids will have improved properties over the former. This is the easiest way for obtaining the intermediate properties. Damping properties of the structure can also be improved by a proper choice of constituent materials.

The objective of a designer is to control the vibration when it is not desirable. If not controlled, this may eventually lead to a failure due to fatigue.

A study of the available open literature reveals that a very few investigations have been made on the dynamic response of hybrid laminated composite plates. In most of the previous investigations either the transverse shear effect was neglected, or the first order shear deformation theory has been employed. Some investigators have used a shear correction factor to take into account the transverse shear effects.

In the present work an attempt has been made to obtain the free vibration characteristics of rectangular hybrid laminated composite plates taking into account shear deformation and rotatory inertia effects. A higher order theory proposed by Reddy[9] has been employed to obtain natural frequencies and mode shapes, using Rayleigh-Ritz technique.

1.2 Literature Review

The available literature which are relevant to the present work can be classified into the following categories:

1. Classical thin plate theory.
2. First order shear deformation theory.
3. Second order shear deformation theory.
4. Higher order shear deformation theory.

In what follows, an attempt has been made to explain how each of these theories have been used in the past for the analysis of isotropic as well as anisotropic plates.

1.2.1 Classical Thin Plate Theory

It was Lekhnitskii[1], who applied the Kirchoff's hypothesis for the analysis of symmetric laminates. He published a book wherein he has employed the single layer anisotropic plate theory based on classical plate theory for systematically analysing the bending, stability and

vibration characteristics of specially orthotropic laminates. A generally orthotropic plate has been analysed using the Kantrovich first iteration method[2].

Reissner and Stavsky[3] improved this analysis by taking into account bending-extensional coupling in the case of unsymmetrically laminated plates.

Whitney and Leissa[4] used this theory for the dynamic analysis of composite plates. They obtained a closed form solution for the linearised equations for the cases of bending, vibration and buckling of some important classes of laminates for which coupling between bending and stretching is present. They found that coupling increased deflections while decreased the fundamental frequency and buckling loads. They concluded that the amount of reduction of the effective stiffness of the plates depends on the degree of anisotropy of the individual layers ~~levels~~ and on the total number of plies in the laminate.

Durvasula and Srinivasan[5] using Huffington and Hoppmann's[6] equations obtained solutions for exact frequency equations for orthotropic plates with one pair of opposite edges simply supported while the other pair of edges supported in any combination. They employed Rayleigh-Ritz technique for the study of vibration of uniform rectangular orthotropic plates with different boundary conditions using beam characteristic functions. They also found out the buckling parameters for a plate under uniform compression using the corresponding frequency parameters.

Lin[7] studied the free vibration characteristics of unsymmetrically laminated plates using classical Kirchoff's assumptions. For more general boundary conditions Bolotin's assymptotic method has been employed and natural frequencies and natural modes of isotropic homogeneous plates and orthotropic rectangular plates with simply supported and clamped edges have been obtained.

Bert and Mayberry[8] used the classical thin lamination theory for the free vibration analysis of rectangular unsymmetrically laminated anisotropic plates with clamped edges. They employed the Rayleigh-Ritz technique and compared the fundamental frequency obtained using this theory with the experimental results. It was found that results were comparable with the experimental results only for the fundamental mode.

Patra and Iyengar[9] studied the free vibration response of laminated rectangular plates with and without cut outs after suitably modifying the thin plate bending element used for isotropic plates. They observed that frequency changes with fibre orientation and that the mode shapes continuously change with fibre orientation.

Umaretiya and Iyengar[10,11] have studied the free vibration characteristics of hybrid laminated composite plates. They have considered rectangular as well as

skew plates with one pair of opposite edges simply supported and the other pair of edges clamped. They employed the Galerkin-Vaslov technique . A linear displacement field has been considered. They have obtained optimal layup details with constraints on maximum deflection and fundamental frequency. In this investigation it was shown that the hybrid laminates give intermediate behaviour between that of single material composites which may be desirable from other considerations.

1.2.2 First Order Shear Deformation Theories

It was Reissner[12], who considered the effect of shear deformation on the bending of elastic plates. Mindlin[13] based on similar theory proposed a displacement field to take into account the effects of transverse shear and rotatory inertia and obtained solutions for the flexural vibration of isotropic elastic plates, with the following displacement field

$$u = Z\Psi_x (x, y, t)$$

$$v = Z\Psi_y (x, y, t)$$

$$w = w (x, y, t)$$

Some investigators have studied the problem employing a shear correction factor.

Yang et al[14] proposed another form of shear deformation theory. They obtained two dimensional theory of motion of heterogeneous plates from three dimensional theory

of elasticity. Transverse shear and rotatory inertia was included in the general theory. They proposed the following form of displacement field,

$$u = u_o(x,y,t) + Z\Psi_x(x,y,t)$$
$$v = v_o(x,y,t) + Z\Psi_y(x,y,t)$$
$$w = w(x,y,t) .$$

They showed that the frequency curves obtained by using classical lamination theory deviated very much from the ones obtained using this theory for anisotropic plates. This theory is commonly known as YNS theory. Whitney and Pagano[15] extended this theory for the static and dynamic analysis of laminated plates consisting of arbitrary number of bonded anisotropic layers.

Levinson[16] and Murthy[17] used the equilibrium equations of the first order shear deformation theory proposed by Whitney and Pagano[15]. Later on Reddy[18] showed that these equations were variationally inconsistent for the displacement field used, with those derived from the principle of virtual displacements.

Bert and Chen[19] using the YNS theory obtained closed form solution for the vibration of antisymmetric angle ply laminated rectangular plates taking into account the effect of shear deformation, parametric effects such as aspect ratio, length to thickness ratio, number of layers

and fibre orientation. Simply supported boundary conditions on all edges are considered. It was found that aspect ratio effect was more pronounced for thick plates than for thin plates and that error in classical plate theory as compared to shear deformation theory increases with an increase in the number of either longitudinal or transverse waves.

Patra and Iyengar[20,21] used a displacement field very much similar to the one used by Mindlin for the free vibration study of composite plates by finite element method. The displacement field through out the plate is as below:

$$u(x,y,z) = -Z\,\phi_x\,(x,y)$$

$$v(x,y,z) = -Z\,\phi_y\,(x,y)$$

$$w(x,y,z) = w_o(x,y)$$

where ϕ_x and ϕ_y are the rotations of the normals and w_o is the transverse displacement of the mid plane. They used a 8-noded, 3 degree of freedom finite elements for this study.

Reddy[22] studied the free vibration of antisymmetric angle ply laminates including transverse shear deformation effects using Finite Element Method. Like Bert and Chen[19] he also used the YNS theory. Non-dimensional frequency parameters for isotropic plates were obtained and compared with the solutions obtained from 3-D linear elasticity.

Results obtained for antisymmetric plies were compared with the closed form solutions of Bert and Chen[19]. It was found that for a given aspect ratio finite element results approached closed form results as side to thickness ratio decreased. A(2x2)mesh of quadratic element was used for quarter plate in all these studies.

Pryor and Barker[23] have used a finite element model including transverse shear deformation effects for arbitrary laminated plates. They have employed a 28 degree freedom discrete element. The displacement formulation used corresponds to first order shear deformation theory. They have used this only for the static analysis of laminated plates and have shown the effect of transverse shear deformation on deflection and stresses in comparison to the classical lamination theory.

Joshi and Iyengar[24] employing finite element technique have carried out the static and dynamic analysis of hybrid laminated composite plates. The shear flexibility is included in the finite element modelling. A nine noded isoparametric element has been employed using Whitney and Pagano's[15] laminate theory wherein transverse shear and rotatory inertia have been taken into account as in Mindlin's[13] theory for isotropic plates.

The non-dimensional frequency and deflection parameters for thick and thin, rectangular and skew hybrid laminated composite plates with and without mixed boundary conditions have been obtained. They found that hybrid laminates provide stiffnesses intermediate to that of single material laminates and that the first natural frequency of the hybrid laminate can be made higher than the stiffer single material laminate.

1.2.3 Second Order Shear Deformation Theory

Whitney and Sun[25] proposed a second order shear deformation theory where in two terms were considered in the series form of the displacement function as follows. The effect of thickness on lateral displacement was also considered.

$$u = u_o(x,y,t) + Z\Psi_x(x,y,t) + \frac{Z^2}{2}\phi_x(x,y,t)$$

$$v = v_o(x,y,t) + Z\Psi_y(x,y,t) + \frac{Z^2}{2}\phi_y(x,y,t)$$

$$w = w_o(x,y,t) + Z\Psi_z(x,y,t)$$

They obtained the fundamental frequency for various laminate layups with simply supported conditions. All these results were compared with those obtained from classical anisotropic theory and the exact results of elasticity solution. It was found that this higher order theory yields results comparable with those of elasticity solution.

1.2.4 Higher Order Shear Deformation Theory

Lo et al[26] developed a higher order theory for analysing the plate deformations. They in addition to the higher order terms considered by Whitney and Sun[25], considered a third order term in the displacement function for inplane displacements and a second order term for the lateral displacement function 'w'. The form of this displacement field is as follows:

$$u = u_o(x,y,t) + Z.\psi_x(x,y,t) + Z^2\zeta_x(x,y,t) + Z^3\phi_x(x,y,t)$$

$$v = v_o(x,y,t) + Z.\Psi_y(x,y,t) + Z^2\zeta_y(x,y,t) + Z^3\phi_y(x,y,t)$$

$$w = w_o(x,y,t) + Z.\Psi_z(x,y,t) + Z^2\zeta_z(x,y,t)$$

They used this higher order theory for the static analysis of laminated composite plates and obtained results comparable with the exact results of elasticity solution.

Similarly Kamal and Durvasula[27,28] presented a higher order shear deformation theory for the dynamic analysis of rectangular and skew composite plates with simply supported and clamped boundary conditions. Here they considered a displacement field such that the condition of zero transverse shear stress at the plate surfaces was satisfied. The displacement field considered is as follows. They did not

consider the effect of thickness on lateral displacement.

$$u = u_o(x,y,t) + f1(Z).\Psi_x + f2(Z) \frac{\partial w}{\partial x}$$

$$v = v_o(x,y,t) + f1(Z).\Psi_y + f2(Z) \frac{\partial w}{\partial x}$$

$$w = w(x,y,t)$$

where

$$f1(Z) = \frac{5}{4} \cdot Z - \frac{5}{3h^2} \cdot Z^3$$

$$f2(Z) = \frac{1}{4} \cdot Z - \frac{5}{3h^2} \cdot Z^3$$

Rayleigh-Ritz technique is employed for obtaining the natural frequencies and the mode shapes of simply supported and clamped, rectangular and skew composite plates. Results so obtained are compared with classical lamination theory and other higher order theories in which Finite Element technique was used. It was found that the results were comparable and that the effect of thickness on lateral displacement was negligible.

Reddy[29] also proposed a higher order theory which is unique in comparison to the earlier mentioned theories. Eventhough, this displacement field very much resembles the one proposed by Kamal and Durvasula[27,28], it differs in one respect. All these earlier mentioned theories were based on assumed displacement field and there was no theoretical backup for these displacement fields. Although

Kamal and Durvasula[27,28] satisfy the condition of zero transverse shear at the laminate surfaces, they assumed a displacement function to satisfy this condition. But Reddy,[29] using the condition of zero transverse shear at laminate surfaces derived the displacement field. The displacement field is as below. A detailed explanation of this model is discussed in Chapter 2.

$$u = u_o(x,y,t) + f1(Z) \cdot \beta_x + f2(Z) \cdot \frac{\partial w}{\partial x}$$

$$v = v_o(x,y,t) + f1(Z) \cdot \beta_y + f2(Z) \cdot \frac{\partial w}{\partial y}$$

$$w = w_o(x,y,t)$$

where

$$f1(Z) = Z - \frac{4}{3h^2} \cdot Z^3$$

$$f2(Z) = -\frac{4}{3h^2} \cdot Z^3$$

It is observed here that this resembles the displacement field assumed in Ref. 27 and 28. He justified the assumption that 'w' is not a function of 'Z', since the transverse normal stress is of the order of $(h/a)^2$ times the inplane normal stress. Navier approach has been employed to obtain exact solutions of the equations obtained by using the principle of virtual displacement for symmetric cross-ply simply supported laminates. Results

for static analysis have been obtained and compared with three-dimensional elasticity solutions of Pagano with simply supported edge conditions under different loading conditions.

1.3 Objective and Scope of the Present Work

The foregoing literature review clearly indicates that very few investigations have been carried out on the static and dynamic analysis of hybrid laminates and most of the studies either neglected the transverse shear deformation and the rotatory inertia effects, or a first order shear deformation theory was used. In this thesis an attempt has been made to analyse the free vibration response of hybrid laminated composite plates. Typical results have been presented for simply supported boundary conditions. The higher order theory proposed by Reddy[29] has been employed to take into account transverse shear and rotatory inertia effects. Rayleigh-Ritz technique has been used to obtain the natural frequencies and mode shapes. The results obtained have been compared with available results in the literature. To check the accuracy of the technique, study has been made for isotropic plates and regular composites. In all these cases the effect of length to thickness ratio, aspect ratio, number of layers in the laminate for symmetric

and antisymmetric angle-plies with different fibre orientations and for symmetric cross-ply laminates have been obtained. The effect of position of a particular lamina in a laminate also has been analysed. As a special case results for a two layer isotropic hybrid laminate has been obtained. This has been accomplished by a programme developed on DEC-1090 system using standard Eigen-value routine and reported in subsequent chapters.

In Chapter 2 mathematical formulation has been discussed in detail. Simply supported boundary condition and other simplifications are also presented.

In Chapter 3, results have been discussed and at the end scope for futurework with conclusion has been included.

CHAPTER 2

FORMULATION AND ANALYSIS

2.1 Shear Deformation Theory

2.1.1 Introduction

The classical lamination theory is based on the Kirchoff's hypothesis which states that,

- Normals to the midplane before deformation remain straight and normal after deformation.
- Stresses acting in the XY-plane dominate the plate behaviour, i.e., σ_z, τ_{xz} and τ_{yz} are negligible.
- Displacements u,v and w are small in comparison to the plate thickness.
- Strains ε_x, ε_y and γ_{xy} are small when compared to unity.
- Rotatory inertia effects are neglected.

It has been well established from experiments that the above assumptions result in the under prediction of deflection and over prediction of natural frequencies. This is due to the fact that the transverse shear strain and rotatory inertia effects are neglected in the classical lamination theory. The transverse shear stresses although small cannot be neglected and can be calculated by

integrating equilibrium equations. A nondimensional analysis has shown that the transverse shear stresses are of the order of (h/a) times the inplane stresses while the transverse normal stresses are of the order of $(h/a)^2$ times the inplane stresses, where 'h' is the thickness of the plate, 'a' is the length of the plate. From this we can assume,

(i) The transverse shear stresses can not be neglected for all sizes of plates.

(ii) Transverse normal stresses can be neglected since the order of magnitude of these stresses in comparison to inplane stresses is very small.

The errors in buckling load, natural frequency etc. are quite high for plates made of composite materials whose ratio of elastic modulus to shear modulus is very large (e.g. Graphite-epoxy, Boron-epoxy). Hence, for analysing the structural behaviour of composite plates one has to consider a theory which takes into account the transverse shear effects.

2.1.2 Higher Order Shear Deformation Theory

The higher order theories have been developed using the kinematic displacement field defined by

$$u(x,y,z,t) = u_o(x,y,t) + z\,\beta_x(x,y,t) + z^2\,\xi_x(x,y,t) + z^3\,\zeta_x(x,y,t) + \ldots.$$

$$v(x,y,z,t) = v_o(x,y,t) + z\,\beta_y(x,y,t) + z^2\,\xi_y(x,y,t) + z^3\,\zeta_y(x,y,t) + \ldots.$$

$$w(x,y,z,t) = w_o(x,y,t) + z\beta_z(x,y,t) + z^2\xi_z(x,y,t) + z^3\zeta_z(x,y,t) + \ldots.. \qquad (2.1)$$

It can be seen that with each power of the thickness coordinate an additional unknown is introduced into each component of total displacement. The equations derived from the Hamilton's principle would become difficult to handle and for many unknowns physical interpretation cannot easily be given. Also these theories do not satisfy the boundary condition that the transverse shear stresses should be zero at plate surfaces. For these reasons, in this work a refined higher order theory proposed by Reddy[29] has been considered. This theory is also based on the displacement field given by equation (2.1) but the specific form is chosen to satisfy the boundary conditions

$$\tau_{xy}\left(z = \pm\frac{h}{2}\right) = \tau_{yz}\left(z = \pm\frac{h}{2}\right) = 0$$

This choice eliminates the need for any shear correction factor and third order distortion of the normal to the midplane is allowed.

We consider a displacement field wherein the displacements along x- and y-directions are expanded as cubic polinomials of the thickness coordinate and displacement in the transverse direction is assumed to be constant through thickness due to the reasons mentioned in Section 2.1.1

$$u(x,y,z,t) = u_o(x,y,t) + z\beta_x(x,y,t) + z^2\xi_x(x,y,t) + z^3\zeta_x(x,y,t)$$

$$v(x,y,z,t) = v_o(x,y,t) + z\beta_y(x,y,t) + z^2\xi_y(x,y,t) + z^3\zeta_y(x,y,t)$$

$$w(x,y,z,t) = w_o(x,y,t) \qquad (2.2)$$

where u_o, v_o and w_o are the midplane displacements, β_x and β_y are the rotations of the normal to the midplane and the functions ξ_x, ξ_y and ζ_x, ζ_y have to be determined from the condition the shear stresses τ_{xz} and τ_{yz} be zero on the top and bottom surfaces of the plate. For orthotropic plates this condition is equivalent to the requirement that the corresponding strains be zero on these surfaces. We have,

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$= \beta_x + 2z\xi_x + 3z^2\zeta_x + \frac{\partial w_o}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$= \beta_y + 2z\xi_y + 3z^2\zeta_y + \frac{\partial w_o}{\partial z} \qquad (2.3)$$

Using the condition that these strains be zero at $z = \pm h/2$, we get

$$0 = \beta_x + h\xi_x + \frac{3h^2}{4}\zeta_x + \frac{\partial w_o}{\partial x} \quad \text{at } z = +h/2$$

$$0 = \beta_x - h\xi_x + \frac{3h^2}{4}\zeta_x + \frac{\partial w_o}{\partial x} \quad \text{at } z = -h/2$$

From these two relations, we get

$$\xi_x = 0 \quad \text{and} \quad \zeta_x = -\frac{4}{3h^2}\left[\beta_x + \frac{\partial w_o}{\partial x}\right] \qquad (2.4a)$$

Similarly using the relation for γ_{yz}, we get

$$\xi_y = 0 \quad \text{and} \quad \zeta_x = -\frac{4}{3h^2}\left[\beta_y + \frac{\partial w_o}{\partial y}\right] \qquad (2.4b)$$

Substituting relations (2.4a) and (2.4b) in equation (2.2), the displacement field becomes

$$u = u_o + \beta_x \cdot f1(z) + \frac{\partial w_o}{\partial x} \cdot f2(z)$$

$$v = v_o + \beta_y \cdot f1(z) + \frac{\partial w_o}{\partial y} \cdot f2(z)$$

$$w = w_o \tag{2.5}$$

where

$$f1(z) = z - \frac{4}{3h^2} \cdot z^3$$

$$f2(z) = -\frac{4}{3h^2} \cdot z^3$$

$$f1(z) = C1 . Z - C2 . Z^3$$

$$f2(z) = -C4 . Z^3 \tag{2.6}$$

where $C1 = 1$ and $C2 = C4 = \frac{4}{3h^2}$.

The displacement field used by Kamal and Durvasula[27,28] shows that only the form of f1(z) and f2(z) changes since they assumed this form to satisfy the condition that τ_{xz} and τ_{yz} to be zero at the surfaces while Reddy[29] obtained the form using these conditions as shown above. f1(z) and f2(z) found in Ref.27 and 28 are as follows:

$$f1(z) = \frac{5}{4} z - \frac{5}{3h^2} z^3$$

$$f2(z) = \frac{1}{4} z - \frac{5}{3h^2} z^3$$

Thus it can be seen that even though cubic variation of the inplane displacements are considered, the number of

dependent variables remains same as in first order shear deformation theory.

2.2 Strain-Displacement Relations

The strains corresponding to displacement field defined by eq. (2.5) are

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} + f1(z) \cdot \frac{\partial \beta_x}{\partial x} + f2(z) \cdot \frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} + f1(z) \cdot \frac{\partial \beta_y}{\partial y} + F2(z) \cdot \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + f1(z) \cdot \left[\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} + 2 \cdot f2(z) \cdot \frac{\partial^2 w}{\partial x \partial y} \right]$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w_o}{\partial x}$$

$$= f1'(z) \cdot \beta_x + f2'(z) \cdot \frac{\partial w_o}{\partial x} + \frac{\partial w_o}{\partial x}$$

$$= f1'(z) \cdot \beta_x + \frac{\partial w_o}{\partial x} [1 + f2'(z)]$$

$$\gamma_{yz} = f1'(z) \cdot \beta_y + \frac{\partial w_o}{\partial y} [1 + f2'(z)] \tag{2.7}$$

Substituting for f1(z) and f2(z) from eq. (2.6), we get the strains as follows:

$$\varepsilon_x = \varepsilon_{11} + z\,[k_{11} + z^2\,k_{12}]$$

$$\varepsilon_y = \varepsilon_{21} + z\,[k_{21} + z^2\,k_{22}]$$

$$\gamma_{yz} = \varepsilon_{41} + z^2 \cdot k_{42}$$

$$\gamma_{xz} = \varepsilon_{51} + z^2 \cdot k_{52}$$

$$\gamma_{xy} = \varepsilon_{61} + z\,[k_{61} + z^2\,k_{62}]$$

$$\varepsilon_{11} = \frac{\partial u_o}{\partial x}\;;\quad k_{11} = C1 \cdot \frac{\partial \beta_x}{\partial x}\;;\quad k_{12} = -\left(C2 \cdot \frac{\partial \beta_x}{\partial x} + C_4 \cdot \frac{\partial^2 w}{\partial x^2}\right)$$

$$\varepsilon_{21} = \frac{\partial v_o}{\partial y}\;;\quad k_{21} = C1 \cdot \frac{\partial \beta_y}{\partial y}\;;\quad k_{22} = -\left(C2 \cdot \frac{\partial \beta_y}{\partial y} + C_4 \cdot \frac{\partial^2 w}{\partial y^2}\right)$$

$$\varepsilon_{41} = \left(C1 \cdot \beta_y + \frac{\partial w_o}{\partial y}\right);\quad k_{42} = -3\left[C2 \cdot \beta_y + C4 \cdot \frac{\partial w_o}{\partial y}\right]$$

$$\varepsilon_{51} = \left(C1 \cdot \beta_x + \frac{\partial w_o}{\partial x}\right);\quad k_{52} = -3\left(C2 \cdot \beta_x + C4 \cdot \frac{\partial w_o}{\partial x}\right)$$

$$\varepsilon_{61} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\;;\quad k_{61} = \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}$$

$$k_{62} = -\left[C2\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2 \cdot C4 \cdot \frac{\partial^2 w}{\partial x \partial y}\right] \tag{2.8}$$

2.3 Stress-strain Relations

A laminated plate consists of a number of orthotropic plies with perfect bonding. Each ply will have principal orthotropic axes at an arbitrary orientation to the reference axes of the plate. If the state of plane stress is assumed, then inplane stress-strain relations can be written as

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_L \\ \varepsilon_T \\ \gamma_{LT} \end{Bmatrix}^k$$

and the transverse shear stress-strain relations can be written as

$$\begin{Bmatrix} \tau_{TZ} \\ \tau_{LZ} \end{Bmatrix}^k = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{TZ} \\ \gamma_{LZ} \end{Bmatrix}^k \qquad (2.9)$$

where,

$$Q_{11} = E_L/(1 - \nu_{LT}\nu_{TL})$$

$$Q_{22} = E_T/(1 - \nu_{LT}\nu_{TL})$$

$$Q_{12} = \nu_{LT} \cdot E_T/(1 - \nu_{LT}\nu_{TL})$$

$$= \nu_{TL} \cdot E_L/(1 - \nu_{LT} \cdot \nu_{TL})$$

$$Q_{66} = G_{LT}$$
$$Q_{44} = G_{TZ}$$
$$Q_{55} = G_{LZ}$$
$$k = \text{kth layer.} \tag{2.10}$$

These coefficients are known as reduced stiffnesses. E,G, ν are the Young's modulus, shear modulus and Poisson's ratio respectively. The subscripts L and T are parallel and perpendicular to the principal orthotropic axes of each ply.

The stress-strain relations for laminated plate consisting of 'n' number of layers at arbitrary orientation to the reference axes are obtained using transformed stiffness matrix.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{2.11}$$

where $\bar{Q}_{ij}$ are given by

$$\bar{Q}_{11} = Q_{11} C^4 + 2(Q_{12} + 2Q_{66})C^2S^2 + Q_{22}S^4$$

$$\bar{Q}_{22} = Q_{11} S^4 + 2(Q_{12} + 2Q_{66})C^2S2 + Q_{22}C^4$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})C^2S^2 + Q_{12}(C^4 + S^4)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})C^3S + (Q_{12} - Q_{22} + 2Q_{66})CS^3$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})CS^3 + (Q_{12} - Q_{22} + 2Q_{66})C^3S$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})C^2S^2 + Q_{66}(C^4 + S^4)$$

$$\bar{Q}_{44} = Q_{44}\cdot C^2 + Q_{55}\cdot S^2$$

$$\bar{Q}_{55} = Q_{44}\cdot S^2 + Q_{55}\cdot C^2$$

$$\bar{Q}_{45} = (Q_{44} - Q_{55})C^5 \qquad (2.12)$$

where, $C = \mathrm{Cos}(\theta)$ and $S = \mathrm{Sin}(\theta)$

and 'θ' is the angle between the reference axis and principal orthotropy axis as shown in Figure 2.1.

2.4 Plate Constitutive Relations

The stresses in a layered composite vary from layer to layer. Hence it calls for a simpler but statically

equivalent system of force and moment system[30] acting on the cross section of the laminate which can be used for exact analysis of these structures. Such a system has been shown in Fig. 2.2(a) and Fig. 2.2(b). The resultant forces and moments acting on the laminate cross section are defined as follows:

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) dz$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) \, dz$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \cdot z \, dz$$

$$(P_x, P_y, P_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \cdot z^3 \, dz$$

$$(R_x, R_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) \cdot z^2 \, dz \qquad (2.13)$$

Using the relations, Eqs. (2.11) and (2.13), we get the modified constitutive relations which can be used in the buckling and vibration of plates and shells.

$$\begin{bmatrix} \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} \\ \begin{Bmatrix} R_y \\ R_x \end{Bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} & \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \\ \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} & \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} \varepsilon_{41} \\ \varepsilon_{51} \end{Bmatrix} \\ \begin{Bmatrix} k_{42} \\ k_{52} \end{Bmatrix} \end{bmatrix} \qquad (2.14)$$

$$\begin{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \\ \\ \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \\ \\ \begin{Bmatrix} P_x \\ P_y \\ P_{xy} \end{Bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} & \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \\ \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} & \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} & \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{61} \end{Bmatrix} \\ \\ \begin{Bmatrix} k_{11} \\ k_{21} \\ k_{61} \end{Bmatrix} \\ \\ \begin{Bmatrix} k_{12} \\ k_{22} \\ k_{62} \end{Bmatrix} \end{bmatrix}$$

where

(2.14b)

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]^k (1, z, z^2, z^3, z^4, z^6)\, dz$$

for i = 1,2,6

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]^k (1, z^2, z^4)\, dz$$

for i,j = 4,5 (2.15)

2.5 Strain Energy and Kinetic Energy Relations

The strain energy of a rectangular plate can be expressed in terms of the stresses and strains as

$$U = \frac{1}{2} \int_0^a \int_0^b \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz}) \, dx \, dy \, dz \tag{2.16}$$

Using the equations (2.11) and (2.14) we get the strain energy relation for the rectangular laminated plate as below:

$$\begin{aligned}
U = \frac{1}{2} \int_0^a \int_0^b \Big\{ & A_{11} \left(\frac{\partial u_o}{\partial x}\right)^2 + 2A_{12} \frac{\partial u_o}{\partial x} \frac{\partial v_o}{\partial y} + 2A_{16} \frac{\partial u_o}{\partial x} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) + A_{22} \left(\frac{\partial v_o}{\partial y}\right)^2 + 2A_{26} \frac{\partial v_o}{\partial y} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) \\
& + A_{44} \left(c_1^2 \beta_y^2 + \left(\frac{\partial w}{\partial y}\right)^2 + 2c_1 \beta_y \frac{\partial w}{\partial y}\right) \\
& + 2A_{45} \left(c_1^2 \beta_x \beta_y + c_1 \beta_x \frac{\partial w}{\partial y} + c_1 \beta_y \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}\right) \\
& + A_{55} \left(c_1^2 \beta_x^2 + \left(\frac{\partial w}{\partial x}\right)^2 + 2c_1 \beta_x \frac{\partial w}{\partial x}\right) \\
& + A_{66} \left(\left(\frac{\partial u_o}{\partial y}\right)^2 + \left(\frac{\partial v_o}{\partial x}\right)^2 + 2 \frac{\partial u_o}{\partial y} \frac{\partial v_o}{\partial x}\right) \\
& + 2B_{11} \left(c_1 \frac{\partial u_o}{\partial x} \frac{\partial x}{\partial x}\right) + 2 B_{12} \left(c_1 \frac{\partial u_o}{\partial x} \frac{\partial \beta_y}{\partial y} + c_1 \frac{\partial v_o}{\partial y} \frac{\partial \beta_x}{\partial x}\right) \\
& + 2B_{16} \left(c_1 \frac{\partial u_o}{\partial x} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + c_1 \frac{\partial \beta_x}{\partial x} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\right) \\
& + 2B_{22} \left(c_1 \frac{\partial v_o}{\partial y} \frac{\partial \beta_y}{\partial y}\right)
\end{aligned}$$

$$+ 2B_{26} \left(c_1 \frac{\partial v_o}{\partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + c_1 \frac{\partial \beta_y}{\partial y} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\right)$$

$$+ B_{66} \left(2c_1 \frac{\partial u_o}{\partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2c_1 \frac{\partial v_o}{\partial x} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right)\right)$$

$$+ c_1^2 \; D_{11} \left(\frac{\partial \beta_x}{\partial x}\right)^2 + 2c_1^2 \; D_{12} \frac{\partial \beta_x}{\partial x} \frac{\partial \beta_y}{\partial y}$$

$$+ 2c_1^2 \; D_{16} \frac{\partial \beta_x}{\partial x} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + D_{22} \; c_1^2 \left(\frac{\partial \beta_y}{\partial y}\right)^2$$

$$+ 2c_1^2 \; D_{26} \frac{\partial \beta_y}{\partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right)$$

$$-6D_{44} \left(c_1 \, c_2 \, \beta_y^2 + c_2 \beta_y \frac{\partial w}{\partial y} + c_1 \cdot c_4 \beta_y \frac{\partial w}{\partial y} + c_4 \left(\frac{\partial w}{\partial y}\right)^2\right)$$

$$-6 \, D_{45} \left(2c_1 c_2 \beta_x \beta_y + c_2 \beta_x \frac{\partial w}{\partial y} + c_1 c_4 \beta_x \frac{\partial w}{\partial y} + c_2 \beta_y \frac{\partial w}{\partial x} + c_1 c_4 \beta_y \frac{\partial w}{\partial x} + 2c_4 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\right)$$

$$-6 \, D_{55} \left(c_1 c_2 \beta_x^2 + c_2 \beta_x \frac{\partial w}{\partial x} + c_1 c_4 \beta_x \frac{\partial w}{\partial x} + c_4 \left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$+ D_{66} \left(c_1^2 \left(\left(\frac{\partial \beta_x}{\partial y}\right)^2 + \left(\frac{\partial \beta_y}{\partial x}\right)^2\right) + 2c_1^2 \frac{\partial \beta_x}{\partial y} \frac{\partial \beta_y}{\partial x}\right)$$

$$- 2E_{11} \left(c_2 \cdot \frac{\partial u_o}{\partial x} \cdot \frac{\partial \beta_x}{\partial x} + c_4 \frac{\partial u_o}{\partial x} \frac{\partial^2 w}{\partial x^2}\right)$$

$$- 2E_{12} \left(c_2 - \frac{\partial u_o}{\partial x} \cdot \frac{\partial \beta_y}{\partial y} + c_4 \frac{\partial u_o}{\partial x} \frac{\partial^2 w}{\partial y^2}\right.$$

$$\left. + c_2 \frac{\partial v_o}{\partial y} \frac{\partial \beta_x}{\partial x} + c_4 \frac{\partial v_o}{\partial y} \frac{\partial^2 w}{\partial x^2}\right)$$

$$-2E_{16} \left(c_2 \frac{\partial u_o}{\partial x} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right)\right.$$

$$+ 2c_4 \frac{\partial u_o}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + c_2 \frac{\partial \beta_x}{\partial x} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)$$

$$\left. + c_4 \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\right)$$

$$-2E_{22} \left(c_2 \frac{\partial v_o}{\partial y} \frac{\partial \beta_y}{\partial y} + c_4 \frac{\partial v_o}{\partial y} \frac{\partial^2 w}{\partial y^2}\right)$$

$$-2E_{26} \left(c_2 \frac{\partial v_o}{\partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2 \cdot c_4 \cdot \frac{\partial v_o}{\partial y} \frac{\partial^2 w}{\partial x \partial y}\right.$$

$$\left. + c_2 \frac{\partial \beta_y}{\partial y} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) + c_4 \frac{\partial^2 w}{\partial y^2} \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\right)$$

$$-E_{66} \left(2c_2 \frac{\partial u_o}{\partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 4c_4 \frac{\partial u_o}{\partial y} \frac{\partial^2 w}{\partial x \partial y}\right.$$

$$\left. + 2c_2 \frac{\partial v_o}{\partial x} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 4c_4 \frac{\partial v_o}{\partial x} \frac{\partial^2 w}{\partial x \partial y}\right)$$

$$-2F_{11} \left(c_1 c_2 \left(\frac{\partial \beta_x}{\partial x}\right)^2 + c_1 c_4 \frac{\partial \beta_x}{\partial x} \frac{\partial^2 w}{\partial x^2}\right)$$

$$-2F_{12} \left(2c_1 c_2 \frac{\partial \beta_x}{\partial x} \frac{\partial \beta_y}{\partial y} + c_1 c_4 \frac{\partial \beta_x}{\partial x} \frac{\partial^2 w}{\partial y^2}\right.$$

$$\left. + 2c_1 c_4 \frac{\partial \beta_y}{\partial y} \frac{\partial^2 w}{\partial x^2}\right)$$

$$-2F_{16}\left(2c_1c_2 \frac{\partial\beta_x}{\partial x}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right) + 2c_1c_4 \frac{\partial\beta_x}{\partial x} \frac{\partial^2 w}{\partial x\partial y}\right.$$

$$\left. + c_1c_4 \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right)\right)$$

$$-2F_{22}\left(c_1c_2 \left(\frac{\partial\beta_y}{\partial y}\right)^2 + c_1c_4 \frac{\partial\beta_y}{\partial y} \frac{\partial^2 w}{\partial y^2}\right)$$

$$-2F_{26}\left(2c_1c_2 \frac{\partial\beta_y}{\partial y}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right) + 2c_1c_4 \frac{\partial\beta_y}{\partial y} \frac{\partial^2 w}{\partial x\partial y}\right.$$

$$\left. + c_1c_4 \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right)\right)$$

$$+ 9F_{44}\left(c_2^2 \beta_y^2 + c_4^2 \left(\frac{\partial w}{\partial y}\right)^2 + 2c_2c_4\beta_y \frac{\partial w}{\partial y}\right)$$

$$+ 18F_{45}\left(c_2^2 \cdot \beta_x \beta_y + c_2c_4 \beta_x \frac{\partial w}{\partial y} + c_2c_4 \beta_y \frac{\partial w}{\partial x}\right.$$

$$\left. + c_4^2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\right)$$

$$+ 9F_{55}\left(c_2^2 \beta_x^2 + c_4^2 \left(\frac{\partial w}{\partial x}\right)^2 + 2c_2c_4 \beta_x \frac{\partial w}{\partial x}\right)$$

$$-2F_{66}\left(c_1c_2 \left(\left(\frac{\partial\beta_x}{\partial y}\right)^2 + \left(\frac{\partial\beta_y}{\partial x}\right)^2\right)\right.$$

$$\left. +2c_1c_2 \frac{\partial\beta_x}{\partial y} \frac{\partial\beta_y}{\partial x} + 2c_1c_4 \frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right)\right)$$

$$+ H_{11}\left(c_2^2 \left(\frac{\partial\beta_x}{\partial x}\right)^2 + c_4^2 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2c_2c_4 \frac{\partial\beta_x}{\partial x} \frac{\partial^2 w}{\partial x^2}\right)$$

$$+ 2H_{12} \left(c_2^2 \frac{\partial \beta_x}{\partial x} \frac{\partial \beta_y}{\partial y} + c_2 c_4 \frac{\partial \beta_x}{\partial x} \frac{\partial^2 w}{\partial y^2} + c_2 c_4 \frac{\partial \beta_y}{\partial y} \frac{\partial^2 w}{\partial x^2} + c_4^2 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}\right)$$

$$+ 2H_{16} \left(c_2^2 \frac{\partial \beta_x}{\partial x} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2c_2 c_4 \frac{\partial \beta_x}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + c_2 c_4 \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2c_4^2 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial y}\right)$$

$$+ H_{22}\left(c_2^2 \left(\frac{\partial \beta_y}{\partial y}\right)^2 + c_4^2 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2c_2 c_4 \frac{\partial \beta_y}{\partial y} \frac{\partial^2 w}{\partial y^2}\right)$$

$$+ 2H_{26} \left(c_2^2 \frac{\partial \beta_y}{\partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2 c_2 c_4 \frac{\partial \beta_y}{\partial y} \frac{\partial^2 w}{\partial x \partial y} + c_2 c_4 \frac{\partial^2 w}{\partial y^2} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) + 2c_4^2 \cdot \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x \partial y}\right)$$

$$+ H_{66} \left(4c_4^2 \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + c_2^2 \left(\left(\frac{\partial \beta_x}{\partial y}\right)^2 + \left(\frac{\partial \beta_y}{\partial x}\right)^2\right) + 2c_2^2 \frac{\partial \beta_x}{\partial y} \frac{\partial \beta_y}{\partial x} + 4c_2 c_4 \frac{\partial^2 w}{\partial x \partial y} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right)\right)\Big) dx\, dy \qquad (2.17)$$

Similarly the kinetic energy of the plate can be written as

$$T = \frac{1}{2} \int_0^a \int_0^b \int_{-h/2}^{h/2} \rho^{(k)} \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 dx\, dy\, dx \qquad (2.18)$$

where $\rho^{(k)}$ is the mass density of the k^{th} layer. Now using the relations for the displacements u, v and w from equation (2.5) we get the kinetic energy relation for the plate as follows:

$$
\begin{aligned}
T = \frac{1}{2} \int_0^a \int_0^b \{ & \rho \left[\left(\frac{\partial u_o}{\partial t}\right)^2 + \left(\frac{\partial v_o}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] \\
& + 2q1 \left[\frac{\partial u_o}{\partial t} \cdot \frac{\partial \beta_x}{\partial t} + \frac{\partial v_o}{\partial t} \cdot \frac{\partial \beta_y}{\partial t} \right] \\
& + 2q2 \left[\frac{\partial u_o}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial v_o}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial y} \right] \\
& + I1 \left[\left(\frac{\partial \beta_x}{\partial t}\right)^2 + \left(\frac{\partial \beta_y}{\partial t}\right)^2 \right] \\
& + I2 \left[\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 + \left(\frac{\partial^2 w}{\partial t \partial y}\right)^2 \right] \\
& + 2I3 \left[\frac{\partial \beta_x}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial \beta_y}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial y} \right] \} \, dx \, dy
\end{aligned}
\tag{2.19}
$$

where P, q, and I are the normal, normal-rotatory and rotatory inertia coefficients given as

$$
P, q1, q2, I1, I2, I3 = \sum_{k=1}^{N} \int_{z_{k-1}}^{Z_k} \rho^{(k)} [1, f1(z), f2(z), f1^2(z), f2^2(z), f1(z).f2(z)] \, dz
\tag{2.20}
$$

2.6 Rayleigh-Ritz Method

Rayleigh's energy method is used to obtain the fundamental frequency parameter of a system for which solution of the eigen-value problem is difficult to obtain. This method gives a higher estimate of the fundamental frequency parameter. The closeness of this value depends on the degree of resemblance of the trial function to the natural mode.

The trial functions which satisfy all the boundary conditions of the problem and are differentiable as many times as the degree of the system are known as 'comparison functions', while those functions that satisfy all the geometric boundary conditions and is differentiable half as many times as the degree of the system are known as 'admissible functions'. Hence the trial functions selected in Rayleigh-Ritz method must belong at least to the class of 'admissible functions'. The trial functions chosen in this method will be of the following form:

$$u(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} \, \emptyset_i(x) \cdot \Psi_j(y) \qquad (2.21)$$

where a_{ij} are the unknown coefficients to be determined.

$\emptyset_i(x)$ and $\Psi_j(y)$ are admissible functions. These functions $\emptyset_i(x)$ and $\Psi_j(y)$ are known as generating sets and the

coefficients a_{ij} are determined so that u(x,y) resembles as close as possible to the natural modes. This is mathematically equivalent to seeking those values of a_{ij} for which Rayleigh's quotient is stationary.

The Rayleigh's quotient can be expressed as

$$R(u) \quad = \quad = \omega^2 = \frac{V_{max}}{T^*} = \frac{N(u)}{D(u)} = \frac{N(a_{ij}), i, j=1,2,\ldots n}{D(a_{ij}), i, j=1,2,\ldots n}$$

where,

V_{max} is the maximum strain energy

T^* is the reference kinetic energy

ω is the fundamental frequency.

It can be seen that the admissible functions can be integrated over the spatial domain involved in strain energy and kinetic energy thus rendering Rayleigh's quotient as a function of the coefficients a_{ij} (i, j = 1,2,...n). The quotient will have stationary value if the variation vanishes

$$\frac{\partial R}{\partial a_{rs}} = \frac{D\left(\frac{\partial N}{\partial a_{rs}}\right) - N\left(\frac{\partial D}{\partial a_{rs}}\right)}{D^2} = 0, \text{ for } r,s=1,2,\ldots n$$

Denoting the value of ω^2 associated with the stationary value of the quotient by Λ, we get

$$\frac{\partial N}{\partial a_{rs}} - \Lambda \frac{\partial D}{\partial a_{rs}} = 0 \qquad (2.22)$$

Here 'N' denotes the strain energy and 'D' denotes the kinetic energy.

Equation (2.22) is used to obtain a general solution to the vibration problem of a laminated plate. We assume a double series form of admissible function for the displacements u_o, v_o, w_o and rotations β_x and β_y as follows:

$$u_o = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \cdot \phi_{um}(\xi) \cdot \Psi_{un}(\eta)$$

$$v_o = \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn} \cdot \phi_{vm}(\xi) \cdot \Psi_{vn}(\eta)$$

$$w_o = \sum_{m=1}^{M} \sum_{n=1}^{N} O_{mn} \cdot \phi_{wm}(\xi) \cdot \Psi_{wn}(\eta)$$

$$\beta_x = \sum_{m=1}^{M} \sum_{n=1}^{N} X_{mn} \cdot \phi_{xm}(\xi) \cdot \Psi_{xn}(\eta)$$

$$\beta_y = \sum_{m=1}^{M} \sum_{n=1}^{N} Y_{mn} \cdot \phi_{ym}(\xi) \cdot \Psi_{yn}(\eta) \qquad (2.23)$$

where ξ and η are the normalised values of x and y given by $\xi = x/a$ and $\eta = y/b$. $\phi_{um}(\xi)$, $\phi_{vm}(\xi)$, $\phi_{wm}(\xi)$, $\phi_{xm}(\xi)$, $\phi_{ym}(\xi)$ are assumed as trigonometric functions for the inplane displacements (u_o and v_o), lateral displacement (w_o) and rotations β_x and β_y) respectively corresponding to the m^{th} mode satisfying the boundary conditions at $\xi = 0$ and $\xi = 1$. Similarly the Ψ-functions corresponds to the n^{th} mode satisfying the boundary conditions at $\eta = 0$ and $\eta = 1$. Thus depending on the boundary condition in each of the ξ and η directions one can choose the

corresponding admissible functions that will satisfy the boundary conditions. The strain energy and kinetic energy expressions will take the following form after equations (2.23) are substituted in relations (2.17) and (2.19).

$$U = \left(\frac{ab}{2}\right) \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{r=1}^{M} \sum_{s=1}^{N} \frac{1}{a^4} \int_0^1 \int_0^1 [A11 \cdot a^2 \cdot C_{mn} \cdot \phi'_{um}(\xi) \cdot$$

$$\Psi_{un}(\eta) \cdot C_{rs} \cdot \phi'_{ur} \cdot (\xi) \cdot \Psi_{us}(\eta) + 2 \cdot A12 \cdot a^2 \cdot \lambda \cdot C_{mn} \cdot \phi'_{um}(\xi) \cdot \Psi_{un}(\eta) \cdot$$

$$\cdot G_{rs} \cdot \phi_{vr}(\xi) \cdot \Psi'_{vs}(\eta) + 2 \cdot A16 \cdot \{ a^2 \cdot \lambda \cdot C_{mn} \cdot \phi'_{um}(\xi) \cdot \Psi_{un}(\eta)$$

$$C_{rs} \cdot \phi_{ur}(\xi) \cdot \Psi'_{us}(\eta) + a^2 \cdot C_{mn} \cdot \phi'_{um}(\xi) \cdot \Psi_{un}(\eta) \cdot G_{rs} \cdot \phi'_{vr}(\xi) \cdot \Psi_{vs}(\eta)$$

$$+ A22 \cdot a^2 \cdot \lambda^2 \cdot G_{mn} \cdot \phi_{vm}(\xi) \Psi'_{vn}(\xi) \cdot G_{rs} \cdot \phi_{vr}(\xi) \cdot \Psi'_{vs}(\eta)$$

$$+ 2A26 \cdot \{ a^2 \cdot \lambda^2 \cdot G_{mn} \cdot \phi_{vm}(\xi) \cdot \Psi'_{vn}(\eta) \cdot C_{rs} \cdot \phi_{rs}(\xi) \cdot \Psi'_{us}(\eta)$$

$$+ a^2 \cdot \lambda \cdot G_{mn} \cdot \phi_{vm}(\xi) \cdot \Psi'_{vn}(\eta) \cdot G_{rs} \cdot \phi'_{vr}(\xi) \cdot \Psi_{vs}(\eta) \}$$

$$+ A_{44} \cdot \{ C_1^2 \cdot a^4 \cdot Y_{mn} \cdot \phi_{ym}(\xi) \Psi_{yn}(\eta) \cdot Y_{rs} \cdot \phi_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ a^2 \cdot \lambda^2 \cdot O_{mn} \cdot \phi_{wm}(\xi) \cdot \Psi'_{wr}(\eta) \cdot O_{rs} \cdot \phi_{wr}(\xi) \cdot \Psi'_{ws}(\eta)$$

$$+ a^3 \lambda \cdot 2 \cdot Y_{mn} \cdot \phi_{mn}(\xi) \cdot \Psi_{yn}(\eta) \cdot O_{rs} \cdot \phi_{wn}(\xi) \cdot \Psi'_{ws}(\eta) \cdot C1$$

$$+ 2 \cdot A45 \cdot \{ C_1^2 \cdot a^4 \cdot X_{mn} \cdot \emptyset_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot Y_{rs} \cdot \emptyset_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ a^3 \lambda \cdot X_{mn} \cdot \emptyset_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot O_{rs} \cdot \emptyset_{wr}(\xi) \cdot \Psi'_{ws}(\eta) \cdot C1$$

$$+ a^3 \cdot Y_{mn} \cdot \emptyset_{ym}(\xi) \cdot \Psi_{yn}(\eta) \cdot O_{rs} \cdot \emptyset'_{wr}(\xi) \cdot \Psi_{ws}(\eta) \cdot C1$$

$$+ a^2 \cdot \lambda \cdot O_{mn} \cdot \emptyset'_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot O_{rs} \cdot \emptyset_{wr}(\xi) \Psi'_{ws}(\eta) \}$$

$$+ A55 \cdot \{ C1^2 \cdot a^4 \cdot X_{mn} \cdot \emptyset_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot X_{rs} \cdot \emptyset_{xr}(\xi) \cdot \Psi_{xs}(\eta)$$

$$+ a^2 \cdot O_{mn} \cdot \emptyset'_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot O_{rs} \cdot \emptyset'_{wr}(\xi) \cdot \Psi_{ws}(\eta)$$

$$+ 2 \cdot C1 \cdot a^3 \cdot X_{mn} \cdot \emptyset_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot O_{rs} \cdot \emptyset'_{wr}(\xi) \Psi_{ws}(\eta) \}$$

$$+ A66 \cdot \{ a^2 \cdot \lambda^2 \cdot C_{mn} \cdot \emptyset_{um}(\xi) \cdot \Psi'_{un}(\eta) \cdot C_{rs} \cdot \emptyset_{ur}(\xi) \cdot \Psi'_{us}(\eta)$$

$$+ a^2 \cdot G_{mn} \cdot \emptyset'_{vm}(\xi) \cdot \Psi_{vn}(\eta) \cdot G_{rs} \cdot \emptyset'_{vr}(\xi) \cdot \Psi_{vs}(\eta)$$

$$+ 2 \cdot a^2 \lambda \cdot C_{mn} \cdot \emptyset_{um}(\xi) \Psi'_{un}(\eta) \cdot G_{rs} \cdot \emptyset'_{vr}(\xi) \cdot \Psi_{vs}(\eta) \}$$

$$+ 2 \cdot B11 \cdot \{ C1 \cdot a^2 \cdot C_{mn} \cdot \emptyset'_{um}(\xi) \Psi_{un}(\eta) \cdot X_{rs} \cdot \emptyset'_{xr}(\xi) \Psi_{xs}(\eta) \}$$

$$+ 2 \cdot B12 \cdot \{ C1 \cdot a^2 \cdot \lambda C_{mn} \cdot \emptyset'_{um}(\xi) \cdot \Psi_{un}(\eta) \cdot Y_{rs} \cdot \emptyset_{yr}(\xi) \cdot \Psi'_{ys}(\eta)$$

$$+ C1 \cdot [a^2 \lambda \, G_{mn} \cdot \emptyset_{vm}(\xi) \cdot \Psi'_{vs}(\eta) \cdot X_{rs} \cdot \emptyset'_{xr}(\xi) \cdot \Psi_{xs}(\eta)] \}$$

$$+ 2 \cdot B16 \cdot \{ C1 \cdot [a^2 \lambda \cdot C_{mn} \cdot \emptyset'_{um}(\xi) \cdot \Psi_{un}(\eta) \cdot X_{rs} \cdot \emptyset_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ a^2 \cdot C_{mn} \cdot \emptyset'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot Y_{rs} \cdot \emptyset'_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ a^2 \cdot \lambda \cdot X_{mn} \cdot \emptyset'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot C_{rs} \cdot \emptyset_{ur}(\xi) \cdot \Psi'_{us}(\eta)$$

$$+ \; a^2 \cdot \; X_{mn} \, \varnothing'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot G_{rs} \cdot \varnothing'_{rs} \cdot \varnothing'_{vr}(\xi) \cdot \Psi_{vs}(\eta) \,] \}$$

$$+ \; 2 B22 \{ \; C1 \; \cdot a^2 \cdot \lambda^2 \cdot G_{mn} \cdot \varnothing_{vm}(\xi) \cdot \Psi'_{vn}(\eta) \cdot Y_{rs} \cdot \varnothing_{yr}(\xi) \cdot {}'_{ys}(\eta) \; \}$$

$$+ 2 \cdot B26 \cdot \{ \; C1 \; [\; a^2 \cdot {}^2 \cdot G_{mn} \cdot \varnothing_{vm} \cdot \varnothing_{vm}(\xi) \cdot \Psi'_{vs}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \; a^2 \cdot \lambda \cdot G_{mn} \cdot \varnothing_{vm}(\xi) \cdot \Psi'_{vs}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ \; a^2 \cdot \lambda^2 \cdot Y_{mn} \cdot \varnothing_{ym}(\xi) \cdot \Psi'_{ys}(\eta) \cdot C_{rs} \cdot \varnothing_{ur}(\xi) \cdot \Psi'_{us}(\eta)$$

$$+ \; a^2 \cdot \lambda \cdot Y_{mn} \cdot \varnothing_{ym}(\xi) \cdot \Psi'_{ys}(\eta) \cdot G_{rs} \cdot \varnothing'_{vr}(\xi) \cdot \Psi_{ys}(\eta) \,] \}$$

$$+ \; 2 \cdot B66 \cdot \{ \; C1 \; [\; a^2 \cdot \lambda^2 \cdot C_{mn} \cdot \varnothing_{um}(\xi) \cdot \Psi'_{un}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \; a^2 \cdot \lambda \cdot C_{mn} \cdot \varnothing_{um}(\xi) \cdot \Psi'_{un}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ \; a^2 \cdot \lambda \cdot G_{mn} \cdot \varnothing'_{vm}(\xi) \cdot \Psi_{un}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \; a^2 \cdot G_{mn} \cdot \varnothing'_{vm}(\xi) \cdot \quad {}_{vn}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta) \,] \}$$

$$+ \; D11 \cdot \{ \; C1^2 \cdot a^2 \cdot X_{mn} \cdot \varnothing'_{xm}(\xi) \, \Psi_{xn}(\eta) \cdot X_{rs} \cdot \varnothing'_{xr}(\xi) \cdot \Psi_{xs}(\eta) \}$$

$$+ \; 2 \cdot D12 \{ \; C1^2 \cdot a^2 \cdot \lambda \cdot X_{mn} \cdot \varnothing'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot Y_{rs} \cdot \varnothing_{yr}(\xi) \, \Psi'_{ys}(\eta) \}$$

$$+ \; 2 \cdot D16 \; \cdot \{ \; C1^2 \cdot [a^2 \cdot \lambda \cdot X_{mn} \cdot \varnothing'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \; a^2 \cdot X_{mn} \cdot \varnothing'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta) \,] \}$$

$+ \; D22 \; . \{ C1^2 \; . \; a^2 \lambda^2 . \; Y_{mn} . \; \emptyset_{ym}(\xi) . \; \Psi'_{yn}(n) \; . \; Y_{rs} . \emptyset_{yr}(\xi) \Psi'_{ys}(\eta) \}$

$+ \; 2.D26 . \{ C1^2 \; \{ a^2 . \lambda^2 . \; Y_{mn} . \; \emptyset_{ym}(\xi) \; . \Psi'_{yn}(\eta) \; . \; X_{rs} . \emptyset_{xr}(\xi) . \Psi'_{xs}(\eta)$

$+ \; a^2 . \lambda . \; Y_{mn} . \; \emptyset_{ym}(\xi) \; . \Psi'_{yn}(\eta) . \; Y_{rs} . \; \emptyset'_{yr}(\xi) \; . \Psi_{ys}(\eta) \;] \}$

$- \; 6.D44 . \{ C1 . C2 . a^4 . \; Y_{mn} . \; \emptyset_{ym}(\xi) \; \Psi_{yn}(\eta) \; . \; Y_{rs} . \; \emptyset_{yn}(\xi) . \Psi_{ys}(\eta)$

$+ \; C2 . a^3 . \lambda . \; Y_{mn} . \; \emptyset_{ym}(\xi) \; . \Psi_{yn}(\eta) \; . \; O_{rs} . \emptyset_{wr}(\xi) \; . \Psi'_{ws}(\eta)$

$+ \; C4 . [\; C1 . a^3 \; . \lambda . \; Y_{mn} . \; \emptyset_{ym}(\xi) \; . \Psi_{yn}(\eta) \; . \; O_{rs} . \emptyset_{wr}(\xi) \; . \Psi'_{ws}(\eta)$

$+ \; a^2 . \lambda^2 \; . \; O_{mn} . \; \emptyset_{wm}(\xi) \; . \Psi'_{wn}(\eta) \; . \; O_{rs} . \; \emptyset_{wr}(\xi) \; . \; \Psi'_{ws}(\eta) \;] \}$

$- \; 6.D45 \; . \{ \; 2C1 . C2 . a^4 . X_{mn} . \; \emptyset_{xm}(\xi) \; . \; \Psi_{xn}(\eta) \; . \; Y_{rs} . \emptyset_{rs}(\xi) \; . \Psi_{ys}(\eta)$

$+ \; a^3 . \lambda . \; X_{mn} . \emptyset_{xm}(\xi) \; . \; _{xn}(\eta) \; . \; O_{rs} . \emptyset_{wm}(\xi) \; . \Psi'_{ws}(\eta) \; .(C2 \; + \; C1 . C4)$

$+ \; a^3 . Y_{mn} . \emptyset_{ym}(\xi) \; . \Psi_{yn}(\eta) \; . \; O_{rs} . \emptyset'_{wr}(\xi) \; . \Psi_{ws}(\eta) \; .(C2 \; + \; C1 . C4)$

$+ \; 2 . a^2 . \lambda . \; O_{mn} . \; \emptyset'_{wm}(\xi) \; . \Psi_{wn}(\eta) \; . \; O_{rs} . \emptyset_{wr}(\xi) \; \Psi'_{ws}(\eta) \; . \; C4 \; \}$

$- \; 6.D55 . \{ \; C1 . C2 . a^4 . \; X_{mn} . \; \emptyset_{xm}(\xi) \; . \Psi_{xn}(\eta) \; . \; X_{rs} . \; \emptyset_{xr}(\xi) \; . \Psi_{xs}(\eta)$

$+ \; C4 . \; a^2 . \; O_{mn} \; \emptyset'_{wm}(\xi) \Psi_{wn}(n) \; . \; O_{rs} . \; \emptyset'_{wr}(\xi) \; . \Psi_{ws}(\eta)$

$+ \; a^3 . \; X_{mn} . \; \emptyset_{xm}(\xi) \; . \; _{xn}(\eta) \; . \; O_{rs} . \emptyset'_{wm}(\xi) \; . \Psi_{ws}(\eta) \; .(C2 + C1 . C4) \; \}$

$$+ D66 . \{ C1^2 . [a^2 . \lambda^2 . X_{mn} . \phi_{xm}(\xi) . \Psi'_{xn}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ a^2 . Y_{mn} . \phi'_{ym}(\xi) . \Psi_{yn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) . \Psi_{ys}(\eta)]$$

$$+ 2 . C1^2 . a^2\lambda . X_{mn} . \phi_{xm}(\xi) . \Psi'_{xn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) . \Psi_{yn}(\eta)$$

$$- 2 . E11 . \{ C2 . a^2 . C_{mn} . \phi'_{um}(\xi) . \Psi_{um}(\eta) . X_{rs} . \phi'_{xr}(\xi) . \Psi_{xs}(\eta)$$

$$+ C4 . a . C_{mn} . \phi'_{um}(\xi) . \Psi_{un}(\eta) . O_{rs} . \phi''_{wr}(\xi) . \Psi_{ws}(\eta) \}$$

$$-2 . E12 . \{ C2 . a^2 . \lambda . C_{mn} . \phi'_{um}(\xi) . \Psi_{un}(\eta) . Y_{rs} . \phi_{yr}(\xi) . \Psi'_{ys}(\eta)$$

$$+ C4 . a . \lambda^2 . C_{mn} . \phi'_{um}(\xi) . \Psi_{un}(\eta) . O_{rs} . \phi_{wr}(\xi) . \Psi''_{ws}(\eta)$$

$$+ C2 . a^2\lambda . G_{mn} . \phi_{vm}(\xi) . \Psi'_{vn}(\eta) . X_{rs} . \phi'_{xr}(\xi) . \Psi_{xs}(\eta)$$

$$+ C4 . a\lambda . G_{mn} . \phi_{vm}(\xi) . \Psi'_{vm}(\eta) . O_{rs} . \phi''_{wr}(\xi) . \Psi_{ws}(\eta) \}$$

$$-2 . E16 . \{ C2 [a^2 . \lambda . C_{mn} . \phi'_{um}(\xi) . \Psi_{un}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ a^2 . C_{mn} . \phi'_{um}(\xi) . \Psi_{un}(\eta) . Y_{rs} . \phi'_{yr}(\xi) . \Psi_{ys}(\eta)$$

$$+ a^2 . \lambda . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . C_{rs} . \phi_{ur}(\xi) . \Psi'_{us}(\eta)$$

$$+ a^2 . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . G_{rs} . \phi'_{vr}(\xi) . \Psi_{vs}(\eta)]$$

$$+ C4 [2 . a\lambda . C_{mn} . \phi'_{um}(\xi) . \Psi_{un}(\eta) . O_{rs} . \phi'_{wr}(\xi) . \Psi'_{ws}(\eta)$$

$$+ \ a.\lambda \ O_{mn}.\ \phi''_{wm}(\xi)\ .\Psi_{wn}(\eta)\ .\ C_{rs}.\ \phi_{ur}(\xi)\ .\Psi'_{us}(\eta)$$

$$+ \ a\ .\ O_{mn}.\ \phi''_{wm}(\xi)\ .\Psi_{wn}(\eta)\ .\ G_{rs}.\ \phi'_{vr}(\xi)\ .\Psi_{vs}(\eta)\ ]\}$$

$$-2E22\ .\{\ C2.\ a^2\ .\lambda^2.\ G_{mn}.\ \phi_{vm}(\xi)\ .\ \Psi'_{vn}(\eta)\ .\ Y_{rs}.\ \phi_{rs}(\xi)\Psi'_{ys}(\eta)$$

$$+ \ C4\ .\ a.\lambda^3\ .\ G_{mn}.\ \phi_{vm}(\xi)\ .\Psi'_{vn}(\eta)\ .\ O_{rs}.\ \phi_{wr}(\xi)\ .\Psi''_{ws}(\eta)\ \}$$

$$-2.E26\ .\{\ C2.[\ a^2.\lambda^2\ .\ G_{mn}.\ \phi_{vm}(\xi)\ .\ \Psi'_{vn}(\eta)\ .\ X_{rs}.\ \phi_{xr}(\xi).\Psi'_{xs}($$

$$+ \ a^2.\lambda.\ G_{mn}.\ \phi_{vm}(\xi)\ .\Psi'_{vn}(\eta)\ .\ Y_{rs}.\ \phi'_{yr}(\xi)\ .\Psi_{ys}(\eta)$$

$$+ \ a^2.\lambda^2\ .\ Y_{mn}.\ \phi_{ym}(\xi)\ .\Psi'_{yn}(\eta)\ .\ C_{rs}.\ \phi_{ur}(\xi)\ .\Psi'_{us}(\eta)$$

$$+ \ a^2.\lambda.\ Y_{mn}.\ \phi_{ym}(\xi)\ .\Psi'_{yn}(\eta)\ .\ G_{rs}.\ \phi'_{vr}(\xi)\ .\Psi_{vs}(\eta)\ ]$$

$$+ \ C4.[\ 2a.\lambda^2\ .\ G_{mn}.\ \phi_{vm}(\xi)\ .\Psi'_{vn}(\eta)\ .\ O_{rs}.\ \phi'_{wr}(\xi)\ .\Psi'_{ws}(\eta)$$

$$+ \ a.\lambda^3\ .\ O_{mn}.\ \phi_{wm}(\xi)\ .\Psi''_{wn}(\eta)\ .\ C_{rs}.\ \phi_{ur}(\xi)\ .\Psi'_{us}(\eta)$$

$$+ \ a.\lambda^2.\ O_{mn}.\ \phi_{wn}(\xi)\ .\Psi''_{wn}(\eta)\ .\ G_{rs}.\ \phi'_{vr}(\xi)\ .\Psi_{vs}(\eta)\ ]\}$$

$$-\ 2.E66.\{\ C2.[\ a^2.\lambda^2\ .\ C_{mn}.\ \phi_{um}.\ \phi_{um}(\xi)\ .\Psi'_{un}(\eta)\ .\ X_{rs}.$$

$$\phi_{xv}(\xi)\ .\ \Psi'_{xs}(\eta)$$

$$+ \ a^2.\lambda.\ C_{mn}.\ \phi_{um}(\xi)\ .\Psi'_{un}(\eta)\ .\ Y_{rs}.\ \phi'_{yr}(\xi)\ .\Psi_{ys}(\eta)$$

$$+ a^2 . \lambda . G_{mn} . \phi'_{vm}(\xi) \, \Psi_{vn}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ a^2 . G_{mn} . \phi'_{vm}(\xi) . \;\; _{vn}(\eta) \qquad . Y_{rs} . \phi'_{yr}(\xi) \, \Psi_{ys}(\eta)$$

$$+ 2 . \bar{C4} . [a . \lambda^2 . C_{mn} . \phi_{um}(\xi) . \Psi'_{un}(\eta) . O_{rs} . \phi'_{wr}(\xi) . \Psi'_{ws}(\eta)$$

$$+ a . \lambda . G_{mn} . \phi'_{vm}(\xi) . \Psi_{vn}(\eta) . O_{rs} . \phi'_{wr}(\xi) . \Psi'_{ws}(\eta)]\}$$

$$-2 . F11 . \{ C1 . [C2 . a^2 . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . X_{rs} . \phi'_{xr}(\xi) \, \Psi_{xs}(\eta)$$

$$+ C4 . a . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . O_{rs} . \phi''_{wr}(\xi) . \Psi_{ws}(\eta)]\}$$

$$-2 . F12 . \{ C1 . [2 . C2 . a^2 . \lambda . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . Y_{rs} . \phi_{yr}(\xi) . \Psi'_{ys}($$

$$+ C4 . a \lambda^2 . X_{mn} . \phi'_{xm}(\xi) . \Psi_{mn}(\eta) . O_{rs} . \phi_{wr}(\xi) . \Psi''_{ws}(\eta)$$

$$+ C4 . a \lambda . Y_{mn} . \phi_{ym}(\xi) . \;\; '_{yn}(\eta) . O_{rs} . \phi''_{wr}(\xi) . \Psi_{ws}(\eta)]\}$$

$$- 2 . F16 . \{ C1 . [2C2 . \{ a^2 . \lambda . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}$$

$$+ a^2 . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) . \Psi_{yn}(\eta) \}$$

$$+ 2C4 . a \lambda . X_{mn} . \phi'_{xm}(\xi) . \Psi_{xn}(\eta) . O_{rs} . \phi'_{wr}(\xi) . \Psi'_{ws}(\eta)$$

$$+ C4 . \{ a . \lambda . O_{mn} . \phi''_{wm}(\xi) . \Psi_{wn}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ a . O_{mn} . \phi''_{wm}(\xi) . \;\; _{wn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) . \Psi_{ys}(\eta) \}]$$

$$-2.F22.\{C1.[C2.a^2.\lambda^2.Y_{mn}.\phi_{ym}(\xi).\Psi'_{yn}(\eta).Y_{rs}.\phi_{rs}(\xi).\Psi'_{ys}(\eta)$$

$$+C4.a\lambda^3.Y_{mn}.\phi_{ym}(\xi).\Psi'_{yn}(\eta).O_{rs}.\phi_{wr}(\xi).\Psi''_{ws}(\eta)]\}$$

$$-2.F26.\{C1.[2.C2.\{a^2\lambda^2.Y_{mn}.\phi_{ym}(\xi)\Psi'_{yn}(\eta).X_{rs}(\xi).\Psi'_{xs}(\eta)$$

$$+a^2\lambda.Y_{mn}.\phi_{ym}(\xi).\Psi'_{yn}(\eta).Y_{rs}.\phi'_{yr}(\xi).\Psi_{ys}(\eta)\}$$

$$+C4\{2a.\lambda^2.Y_{mn}.\phi_{ym}(\xi).\Psi'_{yn}(\eta).O_{rs}.\phi'_{wr}(\xi)\Psi'_{ns}(\eta)$$

$$+a.\lambda^3.O_{mn}.\phi_{wm}(\xi).\Psi''_{wn}(\eta).X_{rs}.\phi_{xr}(\xi).\phi'_{xr}(\xi)$$

$$+a\lambda^2.O_{mn}.\phi_{wm}(\xi).\Psi''_{wn}(\eta).Y_{rs}.\phi'_{yr}(\xi)\Psi_{ys}(\eta)\}]$$

$$+9.F44.[C2^2.a^4.Y_{mn}.\phi_{ym}(\xi).\Psi_{yn}(\eta).Y_{rs}.\phi_{yr}(\xi).\Psi_{ys}(\eta)$$

$$+c_4^2.a^2.\lambda^2.O_{mn}\phi_{wm}(\xi).\Psi'_{wn}(\eta).O_{rs}.\phi_{wr}(\xi)\Psi'_{ws}(\eta)$$

$$+2.C2.C4.a^3.\lambda.Y_{mn}.\phi_{ym}(\xi)\Psi_{yn}(\eta).O_{rs}.\phi_{wr}(\xi)\Psi'_{ys}(\eta)]$$

$$+18F45.[C_2^2.a^4.X_{mn}.\phi_{xm}(\xi)\Psi_{xn}(\eta).Y_{rs}.\phi_{yr}(\xi)\Psi_{ts}(\eta)$$

$$+C2.C4.\{a^3\lambda.X_{mn}.\phi_{xm}(\xi)\Psi_{xn}(\eta).O_{rs}.\phi_{wr}(\xi).\Psi'_{ws}(\eta)$$

$$+a^3.Y_{mn}.\phi_{ym}(\xi)\Psi_{yn}(\eta).O_{rs}.\phi'_{wr}(\xi).\Psi_{ws}(\eta)\}$$

$$+C_4^2.a^2\lambda.O_{mn}.\phi'_{wm}(\xi)\Psi_{wn}(\eta).O_{rs}.\phi_{wr}(\xi).\Psi'_{ws}(\eta)]$$

$$+ 9.F55.[C_2^2 . a^4 \; X_{mn} . \emptyset_{xm}(\xi) . \Psi_{xn}(\eta) . X_{rs} . \emptyset_{xr}(\xi) . \Psi_{xs}(\eta)$$

$$+ C_4^2 . a^2 . O_{mn} . \emptyset'_{wm}(\xi) . {}_{wn}(\eta) . O_{rs} . \Theta'_{wr}(\xi) . \Psi_{ws}(\eta)$$

$$+ 2.C2.C4. a^3 . X_{mn} . \emptyset_{xm}(\xi) . \Psi_{xn}(\eta) . O_{rs} . \emptyset'_{wr}(\xi) . \Psi_{ws}(\eta)]$$

$$- 2.F66. \{C1.C2.[a^2 . \lambda^2 . X_{mn} . \emptyset_{xm}(\xi) . \Psi'_{xn}(\eta) . X_{rs} . \emptyset_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ a^2 . Y_{mn} . \emptyset'_{ym}(\xi) . {}_{yn}(\eta) . Y_{rs} . \emptyset'_{yr}(\xi) . \Psi_{ys}(\eta)]$$

$$+ 2. C1.C2. a^2 . \lambda . X_{mn} . \emptyset_{xm}(\xi) . \Psi'_{xn}(\eta) . Y_{rs} . \emptyset'_{yr}(\xi) \Psi_{ys}(\eta)$$

$$+ 2. C1.C4.[a\lambda^2 . O_{mn} . \emptyset'_{wm}(\xi) . \Psi'_{wn}(\eta) . X_{rs} . \emptyset_{xr}(\xi) \Psi'_{xs}(\eta)$$

$$+ a \lambda . O_{mn} . \emptyset'_{wm}(\xi) . \Psi'_{wn}(\eta) . Y_{rs} . \emptyset'_{yr}(\xi) . \Psi_{ys}(\eta)]\}$$

$$+ H11 . \{C_2^2 \; a^2 . X_{mn} . \emptyset'_{xm}(\xi) . \Psi_{xn}(\eta) . X_{rs} . \emptyset'_{xr}(\xi) . \Psi_{xs}(\eta)$$

$$+ C_4^2 . O_{mn} . \emptyset''_{wm}(\xi) . \Psi_{wn}(\eta) . O_{rs} . \emptyset''_{wr}(\xi) . \Psi_{ws}(\eta)$$

$$+ 2.C2.C4. a . X_{mn} . \emptyset'_{xm}(\xi) . \Psi_{xn}(\eta) . O_{rs} . \emptyset''_{wr}(\xi) . \Psi_{ws}(\eta)\}$$

$$+ 2. H12.\{C_2^2 . a^2 \lambda . X_{mn} . \emptyset'_{xm}(\xi) . \Psi_{xn}(\eta) . Y_{rs} . \emptyset_{yr}(\xi) . \Psi'_{ys}(\eta)$$

$$+ C2.C4.[a\lambda^2 . X_{mn} . \emptyset'_{xm}(\xi) . \Psi_{xn}(\eta) . O_{rs} . \emptyset_{wr}(\xi) . \Psi''_{ws}(\eta)$$

$$+ a.\lambda . Y_{mn} . \emptyset_{ym}(\xi) . \Psi'_{yn}(\eta) . O_{rs} . \emptyset''_{wr}(\xi) . \Psi_{ws}(\eta)]$$

$$+ \ C_4^2 \cdot \lambda^2 \cdot O_{mn} \cdot \varnothing''_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot O_{rs} \cdot \varnothing''_{wr}(\xi) \cdot \Psi_{ws}(\eta) \}$$

$$+ \ 2 . H16 . \{ C_2^2 \cdot [a^2 \cdot \lambda \cdot X_{mn} \cdot \varnothing'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \ a^2 \cdot X_{mn} \cdot \varnothing'_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta)]$$

$$+ \ C2 . C4 [2 . a\lambda . X_{mn} \cdot \varnothing'_{nm}(\xi) \cdot \Psi_{xn}(\eta) \cdot O_{rs} \cdot \varnothing'_{wr}(\xi) \cdot \Psi'_{ws}(\eta)$$

$$+ \ a \ \lambda . O_{mn} \cdot \varnothing''_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \ a . O_{mn} \cdot \varnothing''_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta)]$$

$$+ \ 2C_4^2 \cdot \lambda \cdot O_{mn} \cdot \varnothing''_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot O_{rs} \cdot \varnothing'_{wr}(\xi) \cdot \Psi'_{ws}(\eta) \}$$

$$+ \ H22 \cdot \{ C_2^2 \cdot a^2 \cdot \lambda^2 \cdot Y_{mn} \cdot O_{mn} \cdot \varnothing_{ym}(\xi) \cdot \Psi'_{yn}(\eta) \cdot Y_{rs} \cdot \varnothing_{yr}(\xi) \cdot \Psi'_{ys}(\eta$$

$$+ \ C_4^2 \cdot \lambda^4 \cdot O_{mn} \cdot \varnothing_{wm}(\xi) \cdot \Psi''_{wn}(\eta) \cdot O_{rs} \cdot \varnothing_{wr}(\xi) \cdot \Psi''_{ws}(\eta)$$

$$+ \ 2 . C2 . C4 . \ a . \lambda^3 \cdot Y_m n . \varnothing_{ym}(\xi) \ \Psi'_{yn}(\eta) \cdot O_{rs} \cdot \varnothing_{wm}(\xi) \cdot \Psi''_{wn}(\eta) \}$$

$$+ \ 2 . H26 . [C_2^2 \cdot [a^2 \cdot \lambda^2 \cdot Y_m n . \varnothing_{yn}(\xi) \ \Psi'_{yn}(\eta) \cdot X_{rs} \cdot \varnothing_{xr}(\xi) \cdot \Psi'_{xs}(\eta)$$

$$+ \ a^2 \cdot \lambda \cdot Y_{mn} \cdot \varnothing_{ym}(\xi) \cdot \Psi'_{yn}(\eta) \cdot Y_{rs} \cdot \varnothing'_{yr}(\xi) \cdot \Psi_{ys}(\eta)]$$

$$+ \ C2 . C4 . [\ 2 . \ a . \lambda^2 \cdot Y_{nm} \cdot \varnothing_{ym}(\xi) \ \Psi'_{yn}(\eta) \cdot O_{rs} \cdot \varnothing'_{wr}(\xi) \Psi'_{ws}(\eta)$$

$$+ \ a . \lambda^3 . O_{mn} . \phi_{wm}(\xi) . \Psi''_{wn}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ \ a . \lambda^2 . O_{mn} . \phi_{wm}(\xi) . \Psi''_{wn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) . \Psi_{ys}(\eta) \ ]$$

$$+ 2 . C_4^2 . \lambda^3 . O_{mn} . \phi_{wm}(\xi) . \Psi''_{wn}(\eta) . O_{rs} . \phi'_{wr}(\xi) . \Psi'_{ws}(\eta)$$

$$+ H66 \ \{ 4 . C_4^2 . \lambda^2 . O_{mn} . \phi'_{wm}(\xi) . \Psi'_{wb}(\eta) . O_{rs} . \phi'_{wr}(\xi) \Psi'_{ws}(\eta)$$

$$+ C_2^2 . [a^2 . \lambda^2 . X_{mn} . \phi_{nm}(\xi) . \Psi'_{xn}(\eta) . X_{rs} . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ a^2 . Y_{mn} . \phi'_{ym}(\xi) . \Psi_{yn}(\eta) \ Y_{rs} . \phi'_{yr}(\xi) . \Psi_{yn}(\eta) \ ]$$

$$+ 2 . C_2^2 . a^2 . \lambda . X_{mn} . \phi_{xm}(\xi) . \Psi'_{xn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) \Psi_{ys}(\eta)$$

$$+ 4 . C2 . C4 . a \lambda^2 . O_{mn} . \phi'_{wm}(\xi) \Psi'_{wn}(\eta) . X_{rs} . . \phi_{xr}(\xi) . \Psi'_{xs}(\eta)$$

$$+ 4 . C2 . C4 . a \lambda . O_{mn} . \phi'_{wm}(\xi) \Psi'_{wn}(\eta) . Y_{rs} . \phi'_{yr}(\xi) \Psi_{ys}(\eta) \}] \, d\xi \, d$$

(2.24)

The kinetic energy given by equation (2.19) now takes the following form:

$$(P,\ q1, q2,\ I1, I2, I3\ ) = \sum_{K=1}^{NL} \int_{z_{K-1}}^{z_K} \rho^{(k)} [\ 1, f1(z),\ f2(z),\ f1^2(z),\ f2^2(z),\ f1(z),\ f2(z)\ ] dz$$

$$\therefore\ T = \frac{1}{2} \int_o^a \int_o^b [P\ (\frac{\partial u_o}{\partial t})^2 + (\frac{\partial v_o}{\partial t})^2 + (\frac{\partial w}{\partial t})^2$$

$$+ 2q1\ \frac{\partial u_o}{\partial t} \cdot \frac{\partial x}{\partial t} + \frac{\partial v_o}{\partial t} \cdot \frac{\partial y}{\partial t}$$

$$+ 2q1\ \frac{\partial u_o}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial v_o}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial y}$$

$$+ I1\ (\frac{\partial x}{\partial t})^2 + (\frac{\partial y}{\partial t})^2$$

$$+ I2\ (\frac{\partial^2 w}{\partial t \partial x})^2 + (\frac{\partial^2 w}{\partial t \partial y})^2$$

$$+ 2I3\ \frac{\partial x}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial y}{\partial t} \cdot \frac{\partial^2 w}{\partial t \partial y}]\ dx\ dy$$

$$T = (\frac{ab}{2}) [\ \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{r=1}^{M} \sum_{s=1}^{N} \int_o^1 \int_o^1 P\ C_{mn} \cdot \emptyset_{um}(\xi) \Psi_{un}(\eta) . C_{rs} \cdot \emptyset_{ur}(\xi) \cdot \Psi_{us}(\eta)$$

$$+ G_{mn} \cdot \emptyset_{vm}(\xi) \cdot \Psi_{vn}(\eta) \cdot G_{rs} \cdot \emptyset_{vr}(\xi)\ \Psi_{vs}(\eta) + O_{mn} \cdot \emptyset_{wm}(\xi) \cdot \Psi_{wn}(\eta)$$

$$O_{rs} \cdot \dot{\emptyset}_{wr}(\xi) \cdot \ _{ws}(\eta)$$

$$+ \; 2q1 \quad C_{mn} \cdot \emptyset_{um}(\xi) \cdot \Psi_{un}(\eta) \cdot X_{rs} \cdot \emptyset_{xr}(\xi) \cdot \Psi_{xs}(\eta)$$

$$+ \; G_{mn} \cdot \emptyset_{vm}(\xi) \cdot \Psi_{vn}(\eta) \cdot Y_{rs} \cdot \emptyset_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ \; 2q2 \quad C_{mn} \cdot \emptyset_{um}(\xi) \; \Psi_{un}(\eta) \cdot O_{rs} \cdot \frac{1}{a} \cdot \emptyset'_{wr}(\xi) \cdot \Psi_{ws}(\eta)$$

$$+ \; G_{mn} \cdot \emptyset_{vm}(\xi) \cdot {}_{vm}(\eta) \cdot O_{rs} - \frac{1}{b} \cdot \emptyset_{wr}(\xi) \; \Psi'_{ws}(\eta)$$

$$+ \; I1 \quad X_{mn} \cdot \emptyset_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot X_{rs} \cdot \emptyset_{xr}(\xi) \cdot \Psi_{xs}(\eta)$$

$$+ \; Y_{mr} \cdot \emptyset_{ym}(\xi) \cdot \Psi_{yn}(\eta) \cdot Y_{rs} \cdot \emptyset_{yr}(\xi) \cdot \Psi_{ys}(\eta)$$

$$+ \; I2 \quad \frac{1}{a^2} \cdot O_{mn} \cdot \emptyset'_{wm}(\xi) \cdot \Psi_{wn}(\eta) \cdot O_{rs} \cdot \emptyset'_{wr}(\xi) \cdot \Psi_{ws}(\eta)$$

$$+ \; \frac{1}{b^2} \cdot O_{mn} \cdot \emptyset_{wm}(\xi) \cdot \Psi'_{wn}(\eta) \cdot O_{rs} \cdot \emptyset_{wr}(\xi) \cdot \Psi'_{ws}(\eta)$$

$$+ \; I3 \quad \frac{1}{a} \cdot X_{mn} \cdot \emptyset_{xm}(\xi) \cdot \Psi_{xn}(\eta) \cdot O_{rs} \cdot \emptyset'_{wr}(\xi) \cdot \Psi_{ws}(\eta)$$

$$+ \; \frac{1}{b} \cdot Y_{mn} \cdot \emptyset_{ym}(\xi) \cdot \Psi_{yn}(\eta) \cdot O_{rs} \cdot \emptyset_{wr}(\xi) \; \Psi'_{ws}(\eta)] \quad d\xi \, d\eta \qquad (2.25)$$

Now using the equation (2.22) and the notations for integrals given in Appendix-A, we get a set of linear homogeneous, algebraic relations which can be concisely put in matrix form as follows:

$$\left\{ [S] - K_n^2 [M] \right\} \begin{Bmatrix} C_{rs} \\ G_{rs} \\ O_{rs} \\ X_{rs} \\ Y_{rs} \end{Bmatrix} = 0 \qquad (2.26)$$

This is a standard eigen-value problem, where K_n is the non-dimensional frequency parameter.

$$K_n = \omega_n a^2 \sqrt{\frac{\rho}{E_T h^2}} \qquad (2.27)$$

[S]is the stiffness matrix and[M]is the mass matrix the elements of which are as follows.

Stiffness Matrix Elements

$$S11 = a^2 \left[A11 . I1mr^{11} . J1ns^{00} + 2\, A16 \lambda . I1mr^{10}\ J1ns^{01} \right.$$
$$\left. + A66 . \lambda^2 . I1mr^{00}\ J1.ns^{11} \right]$$

$$S12 = 2\, a^2 \left[A12 . \lambda . I6mr^{10} . J6ns^{01} + A16\ I6mr^{11}\ J6ns^{00} \right.$$
$$\left. + A26 . \lambda^2 . I6mr^{00} . J6ns^{11} + A66 . \lambda . I6mr^{01}\ J6ns^{10} \right]$$

$$S13 = -2a \left[I7mr^{12}\ J7ns^{00}\ C4E11 + \lambda^2\ I7mr^{10}\ J7ms^{02} . C4 . E12 \right.$$
$$+ C4 \lambda\, E16\ (2 I7mr^{11}\ J7ns^{01} + I7\ mr^{02} . J7ns^{10})$$

$$+\lambda^{3}\cdot C4\cdot E26\,(I7mr^{00}\cdot J7ns^{12})+\lambda^{2}\cdot I7mr^{01}\cdot J7ns^{11}\cdot C4\cdot E66\ ]$$

$$S14 = 2a^{2}\,[I8mr^{11}\cdot J8ns^{00}\ (C1B11 - C2\ E11)$$

$$+\ 2\lambda\, I8mr^{01}\ J8ns^{10}\ (C1B16 - C2\ E16)$$

$$+\ \lambda^{2}\ I8mr^{00}\ J8ns^{11}\ (C1\ B66 - C2\ E66)\,]$$

$$S15 = 2a^{2}[\lambda\cdot I9mr^{10}\ J9ns^{01}\ (C1B12 - C2\cdot E12)$$

$$+\ I9mr^{11}\ J9ns^{00}\ (C1B16 - C2\cdot E16)$$

$$+\ \lambda^{2}\ I9mr^{00}\ J9ns^{11}\ (C1B26 - C2\ E26)$$

$$+\ \lambda\, I9mr^{01}\ J9ns^{10}\ (C1\ B66 - C2\cdot E66)\ ]$$

$$S22 = a^{2}\,[\,A22\,\lambda^{2}\ I2mr^{00}\ J2ns^{11} + 2A26\,\lambda\ I2mr^{01}\ Jrns^{10}$$

$$+\ A66\ I2mr^{11}\ J2ns^{00}\,]$$

$$S23 = -2a\,[\lambda I10mr^{02}\ J10ns^{10}\ C4\ E12$$

$$+\ I10mr^{12}\ J10ns^{00}\ C4\ E16 + \lambda^{3}\ I10mr^{00}\ J10ns^{12}\ C4\cdot E22$$

$$+\ \lambda^{2}\ C4\ E26\,(2\ I10mr^{01}\cdot J10ns^{11} + I10mr^{10} + J10ns^{02})$$

$$+\ 2\cdot\lambda\cdot\ I10mr^{11}\ J10ns^{01}\ C4\cdot E66\ ]$$

$$S24 = 2a^2 [\lambda . I^{01}_{11mr} \; J^{10}_{11ns} \; (C1 \; B12 - C2 . E12)$$

$$+ \; I^{11}_{11mr} \; J^{00}_{11ns} \; (C1 \, . \, B16 - C2 . \, E16)$$

$$+ \; \lambda^2 \; I^{00}_{11mr} \; J^{11}_{11ns} \; (C1 \; B26 - C2 \; E26)$$

$$+ \lambda . I^{10}_{11mr} \; J^{01}_{11ns} \; (C1 \; B66 - C2 . \, E66)$$

$$S25 = 2a^2 [\lambda^2 \; I^{00}_{12mr} \; J^{11}_{12ns} \; (C1 . \, B22 - C2 . \, E22)$$

$$+ \; 2 . \lambda . I^{10}_{12mr} \; J^{01}_{12ns} \; (C1 . B26 - C2 . E26)$$

$$+ \; I^{11}_{12mr} \; J^{00}_{12ns} \; (C1 . \, B66 - C2 . \, E66) \;]$$

$$S33 = a^2 \lambda^2 \; I^{00}_{3mr} \; J^{11}_{3ns} \; [\, A44 - 6D44C4 + 9 . F44 \; C4^2 \,]$$

$$+ \; 2a^2 . \lambda . I^{10}_{3mr} \; J^{01}_{3ns} \; [A45 - 6 \; D45 \, . \, C4 + 9 \; F45 \; C4^2 \,]$$

$$+ \; a^2 \; I^{11}_{3mr} \; J^{00}_{3ns} \; [A55 - 6D55 \; C4 + 9F55 \; C4^2 \,]$$

$$+ \; I^{22}_{3mr} \; J^{00}_{3ns} H_{11} C4^2 + 2 . \lambda .^2 \; I^{20}_{3mr} \; J^{02}_{3ns} \; H12 . \, C4^2$$

$$+ \; 4 . \lambda . I^{21}_{3mr} \; J^{01}_{3ns} \; H16 \; C4^2 + \lambda^4 . \, I^{00}_{3mr} \; J^{22}_{3ns} \; H22 . \, C4^2$$

$$+ 4\lambda^{3}\, I3_{mr}^{01}\, J3_{ns}^{21}\, H26\, C4^{2} + 4\lambda^{2}\, I3_{mr}^{11}\, J3_{ns}^{11}\, H66\, C4^{2}$$

$$S34 = 2\, a^{3}\lambda\, I13_{mr}^{00}\, J13_{ns}^{01} [C1\; A45 - 3\, D45\, (C2 + C1\cdot C4)$$

$$+ 9F45\, C2\cdot C4]$$

$$+ 2a^{3}\, I13_{mr}^{01}\, J13_{ns}^{00} [A55\; C1 - 3D55\, (C2 + C1C4)$$

$$+ 9\, F55\, C2\cdot C4]$$

$$+ 2a\, I13_{mr}^{12}\, J13_{ns}^{00} [C2\cdot C4\cdot H11 - C1\cdot C4\cdot F11]$$

$$+ 2a\cdot\lambda^{2}\cdot I13_{mr}^{10}\, J13_{ns}^{02} [(C2\cdot C4\cdot H12 - C1\cdot C4\cdot F12)]$$

$$+ 2a\cdot\lambda(2\, I13_{mr}^{11}\, J13_{ns}^{01} + I13_{mr}^{02}\, J13_{ns}^{10})(C2\cdot C4\cdot H16$$

$$- C1\cdot C4\cdot F16)$$

$$+ 2a\lambda^{3}\, I13_{mr}^{00}\, J13_{ns}^{12}\, (C2C4\; H26 - C1\cdot C4\cdot F26)$$

$$+ 4a\lambda^{2}\, I13_{mr}^{01}\, J13_{ns}^{11}\, (C2\cdot C4\cdot H66 - C1\cdot C4\cdot F66)$$

$$S_{35} = 2a^{3}\cdot\lambda\cdot I14_{mr}^{00}\, J14_{ns}^{01}\, (C1\cdot A44 - 3(C2 + C1\cdot C4)\, D44$$

$$+ 9C2\cdot C4\; F44) + 2a^{3}\, I14_{mr}^{01}\, J14_{ns}^{00}\, (C1\; A45 - 3\, (C2+C1C4)$$

$$D45 + 9C2\cdot C4F45)$$

$$+ 2\lambda a \; I14mr^{02} \; J14ns^{10} \; (C2.C4.H12 - F12.C1.C4)$$

$$+ 2a \; I14\,mr^{12} \; J14\,ns^{00} \; (C2.C4.H16 - F16\; C1.C4)$$

$$+ 2a\lambda^{3} \; I14mr^{00} \; J14ns^{12} \; (C2.C4\; H22 - C1.C4\; F22)$$

$$+ 2a\lambda^{2} \; (C2.C4.H26 - C1.C4\; F26)(2\; I14mr^{01} \; J14ns^{11} + I14mr^{10} \; J14ns^{02})$$

$$+ 4a\lambda \; I14mr^{11} \; J14ns^{01} \; (C2.C4\; H66 - C1.C4\; F66)$$

$$S44 = a^{4} \; I4mr^{00} \; J4ns^{00} \; (C1^{2}.\; A55 - 6C1.C2\; D55 + 9C2^{2}\; F55)$$

$$+ a^{2} \; I4mr^{11} \; J4ns^{00} \; C1^{2}\; D11 - 2F11.\; C1.C2 + C2^{2}\; H11$$

$$+ 2a^{2}\lambda \; I4mr^{10} \; J4ns^{01} \; C1^{2}\; D16 - 2C1C2\; F16 + C2^{2}\; H16$$

$$+ a^{2}\lambda^{2} \; I4mr^{00} \; J4ns^{11} \; (C1^{2}\; D66 - 2C1C2.F66 + C2^{2}\; H66)$$

$$S45 = 2a^{4} \; I15mr^{00} \; J15ns^{00} \; (C1^{2}\; A45 - C1.C2.D45 + 9C2^{2}\; F45)$$

$$+ 2a^{2}\lambda \; I15mr^{10} \; J15ns^{01} \; (C1^{2}\; D12 - 2C1.C2\; F12 + C2^{2}\; H12)$$

$$+ 2a^{2} \; I15mr^{11} \; J15ns^{00} \; (C1^{2}\; D16 - 2C1C2.F16 + C2^{2}\; H16)$$

$$+ 2a^{2}\lambda^{2} \; I15mr^{00} \; J15ns^{11} \; (C1^{2}\; D26 - 2C1C2.F26 + C2^{2}H26)$$

$$+ 2a^{2}\lambda \; I15mr^{01} \; J15ns^{10} \; (C1^{2}\; D66 - 2C1.C2\; F66 + C2^{2}H66)$$

$$S55 = a^4 \; I5mr^{00} \; J5ns^{00} \; (C1^2 \; A44 - 6C1 \cdot C2 \; D44 + 9C2^2 \; F44)$$

$$+ \; a^2\lambda^2 \; I5mr^{00} \; J5ns^{11} \; (C1^2 \cdot D22 - 2C1 \cdot C2F22 + C2^2 H22)$$

$$+ \; 2a^2 \lambda \; I5mr^{01} \; J5ns^{10} \; (C1^2 \; D26 - 2C1 \cdot C2 \cdot F26 + C2^2 \; H26)$$

$$+ \; a^2 \; I5mr^{11} \; J5ns^{00} \; (C1^2 \cdot D66 - 2C1 \cdot C2 \; F66 + C2^2 \; H66)$$

<u>Mass Matrix Elements</u>

$$M11 = P \cdot I1mr^{00} \cdot J1ns^{00}$$

$$M12 = 0$$

$$M13 = 2q2/a \cdot I7mr^{01} \cdot J7ns^{00}$$

$$M14 = 2q1 \cdot I8mr^{00} \cdot J8ns^{00}$$

$$M15 = 0$$

$$M22 = P \cdot I2mr^{00} \cdot J2ns^{00}$$

$$M23 = 2q2/b \cdot I10mr^{00} \cdot J10ns^{01}$$

$$M24 = 0$$

$$M25 = 2q1 \cdot I12mr^{00} \cdot J12ns^{00}$$

$$M33 = P \cdot I3mr^{00} \cdot J3ns^{00} + I2 \; (1/a^2 \; I3mr^{11} \; J3ns^{00} + 1/b^2 \cdot I3mr^{00} \; J3ns^{11})$$

$$M34 = 2I3/a \cdot I13mr^{01} \; J13ns^{00}$$

$$\beta_x(\xi, 0) = \beta_x(\xi, 1) = 0$$

$$\beta_y(0,\eta) = \beta_y(1,\eta) = 0 \qquad (2.28a)$$

These boundary conditions have been used only for crossply plates by Reddy[31]. While for angle ply laminates following boundary conditions have been employed:

$$u_o, (0,\eta) = u_o(1,\eta) = 0$$

$$v_o(\xi, 0) = v_o(\xi, 1) = 0$$

$$w_o(\xi, 0) = w_o(\xi, 1) = w_o(0, \eta) = w_o(1,\eta) = 0$$

$$\beta_x(\xi, 0) = \beta_x(\xi, 1) = 0$$

$$\beta_y(0,\eta) = \beta_y(1,\eta) = 0 \qquad (2.28b)$$

But in the present work we have employed the same boundary conditions as given in equation (2.28a) for all types of laminate constructions. In Figures 2.3a and 2.3b these boundary conditions have been shown schematically.

2.8 Admissible Functions

As mentioned in Section 2.6, an admissible function should satisfy atleast the geometric boundary conditions and should be differentiable half as many times as the degree of the system. The boundary conditions specified in

the above section can be satisfied by assuming the following form for the admissible functions.

$$\phi_{um}(\xi) = \cos m\pi(\xi) \; ; \; \Psi_{un}(\eta) = \sin n\pi(\eta)$$

$$\phi_{vm}(\xi) = \sin m\pi(\xi) \; ; \; \Psi_{vn}(\eta) = \cos n\pi(\eta)$$

$$\phi_{wm}(\xi) = \sin m\pi(\xi) \; ; \; \Psi_{wn}(\eta) = \sin n\pi(\eta)$$

$$\phi_{xm}(\xi) = \cos m\pi(\xi) \; ; \; \Psi_{xn}(\eta) = \sin n\pi(\eta)$$

$$\phi_{ym}(\xi) = \sin m\pi(\xi) \; ; \; \Psi_{yn}(\eta) = \cos n\pi(\eta)$$

Similarly for other boundary condition suitable admissible functions must be chosen. By looking at the expressions for the stiffness and mass matrix elements, it can be observed that, one will have to obtain corresponding integral values for that boundary condition. Thus this method can be extended to obtain results for some other boundary conditions by using corresponding integrals.

2.9 Mode Shapes

After solving the eigenvalue problem of equation (2.26) the deflected shape of the plate corresponding to a particular mode of lateral vibration can be obtained by substituting the values of the coefficients O^{i}_{mn} in equation (2.23) where i indicates the mode of vibration. The lateral deflection at any point $P(\xi_k, \eta_l)$ is given by

$$W^{i}(\xi_k \eta_l) = \sum_{m=1}^{M} \sum_{n=1}^{N} O^{i}_{mn} \cdot \phi_{wm}(\xi_k) \cdot \psi_{wn}(\eta_l)$$

where the admissible functions will take up a form so as to satisfy the boundary conditions. Here the subscripts k and l denote the point in the field. Points are selected at intervals of Δ k and Δ l which will depend on the plate aspect ratio a/b. The ratio of $\Delta k/\Delta l$ is always made equal to the plate aspect ratio. Using these values of lateral deflection nodal lines can be plotted. To get the exact mode shapes for higher modes one should consider higher number of terms in the series. Moreover depending on the aspect ratio of the plate number of terms in each direction should be varied to get proper convergence.

Thus we have formulated a general solution for the problems of free vibration of laminated composite plates. This can also be used for the static analysis of laminated as well as isotropic plates. This analysis is applicable to any combination in-plane and out-of-plane boundary conditions. In the next chapter numerical computations have been done to show the convergence of the solution, effect of aspect ratio, fibre orientation, number of layers, span to thickness ratio for isotropic, regular as well as hybrid composites. Nodal lines have been obtained for some representative cases.

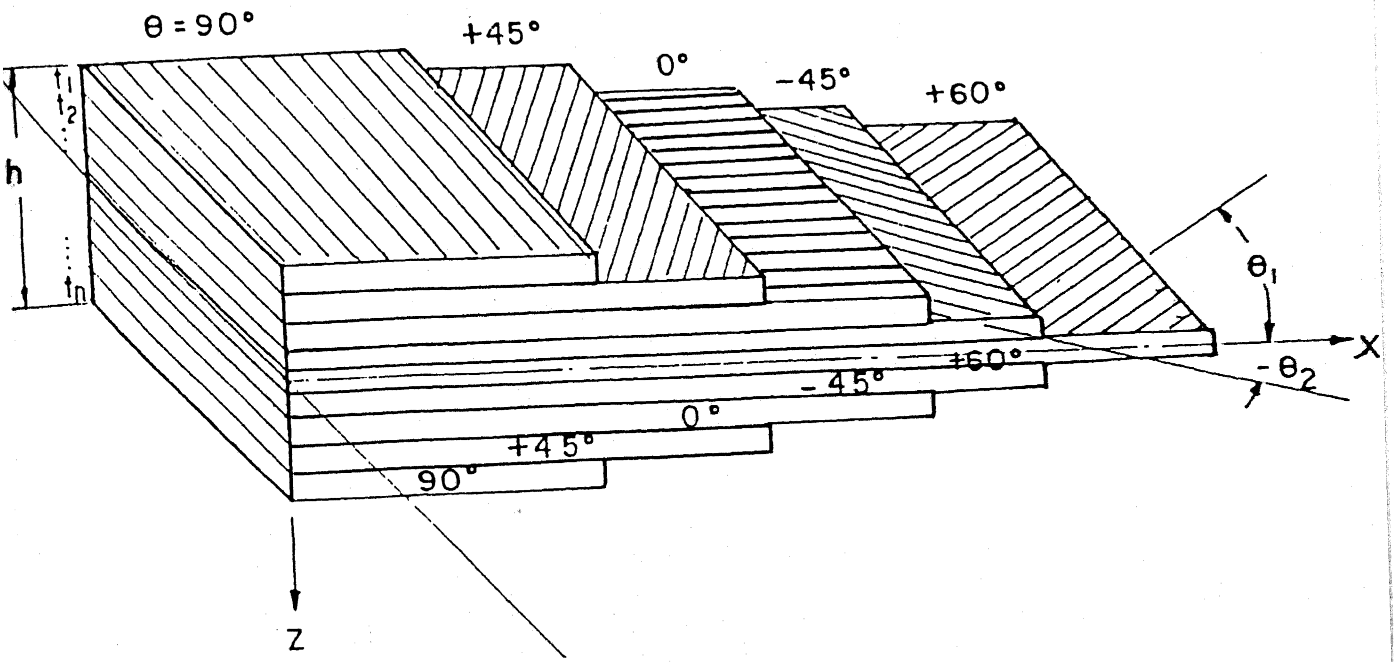

Fig: 2.1 Schematic view of laminate construction and general coordinate system

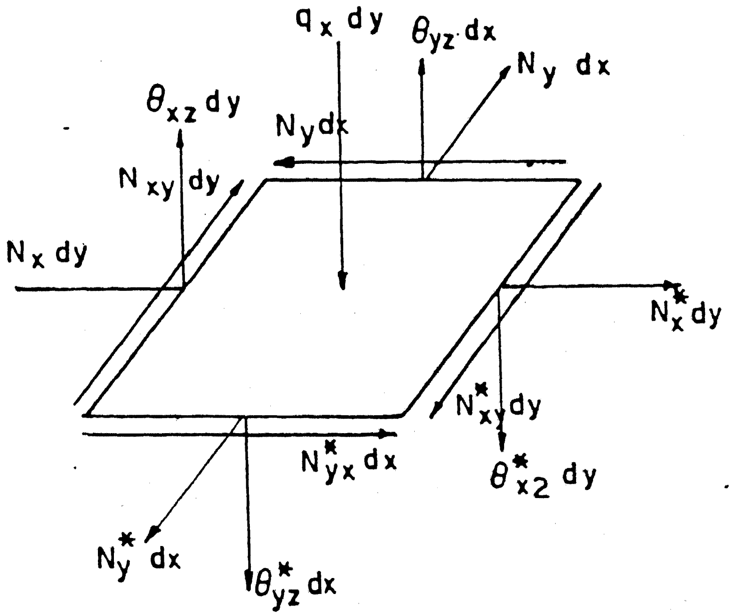

Fig: 2·2a Statically equivalent force system on the laminates associated with membrane action

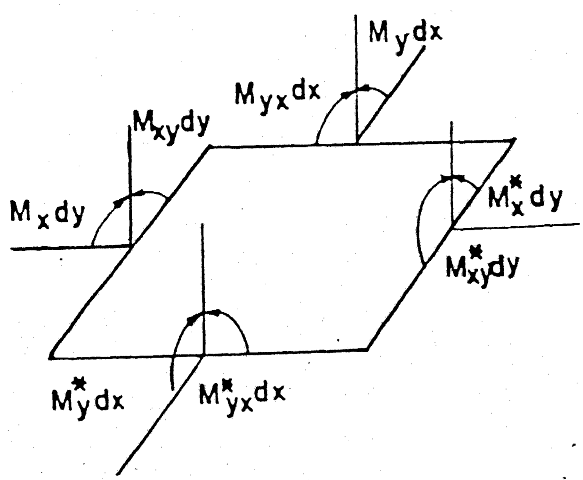

Fig: 2·2b Statically equivalent moment system on the laminate due to bending

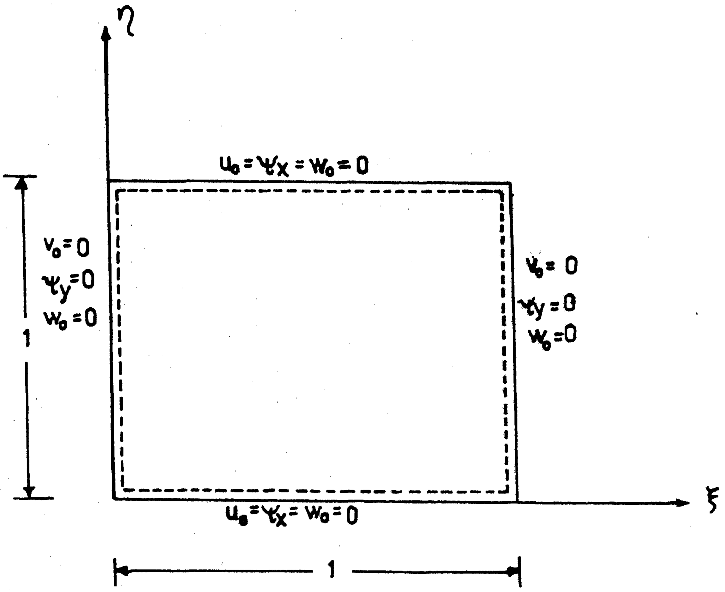

Fig:2.3a Schematic representation of the boundary conditions considered in the present analysis

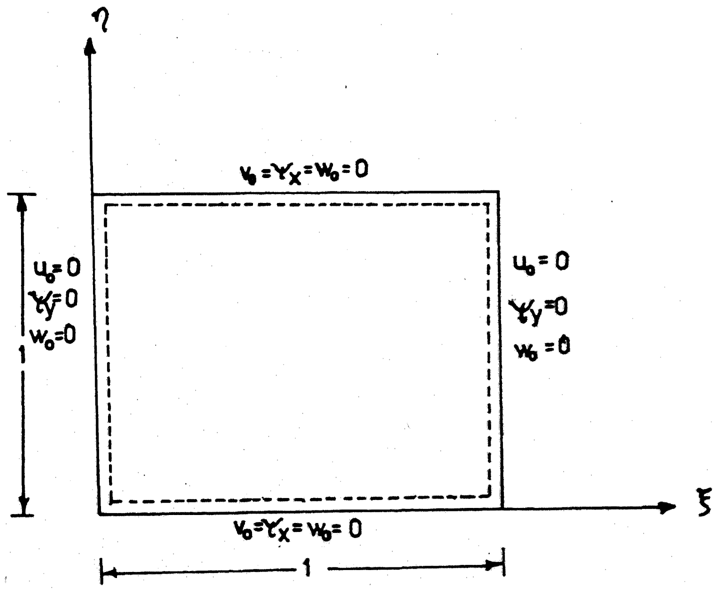

Fig:2.3 b Schematic representation of the boundary conditions considered in Ref. [31]

CHAPTER 3

RESULTS AND DISCUSSION

3.1 Numerical Calculations

The present formulation which is developed for the analysis of hybrid laminated composite plates has been checked by comparing the results for Isotropic and regular composite plates for which results are available in the literature. All these computations have been carried out on DEC-1090 system using standard routines available for numerical integration and Eigenvalue solution. In all these cases M = 4 and N = 4 has been considered in the summation series of the displacement and rotation functions. This will lead to 16 terms in each of the displacement and rotation functions thus rendering the overall stiffness matrix [S] and mass matrix [M] to be of the order of (80x80). A convergence study has been carried out and it has been found that M = N = 4 is the optimal value for which satisfactory convergence can be achieved. Beyond this the storage space required and CPU time becomevery high and from the point of convergence it has been found that there is no appreciable change in the numerical values obtained.

Computer programme thus generated is later on used for the parametric studies on the behaviour of hybrid laminated

composite plates. Three types of materials have been considered and non-dimensional mechanical properties of these materials areshown in Table 3.1. For the hybrid laminate GFRP fibre having higher E_L/E_T ratio has been considered since this gives better properties than the second type of GFRP for which $E_L/E_T = 25$.

For the purposes of this investigation only four layers are considered. Numerical results have been obtained for a hybrid laminate in which Graphite/Epoxy plies form the outer layers while Kevlar/Epoxy plies form the inner layers to study the effect of various parameters such as aspect ratio, length to thickness ratio of the plate, fibre orientation and number of layers. Shear deformation effects have been studied by considering thick plates as well as thin plates. In what follows we shall discuss effect of each of these parameters on natural frequencies.

3.2 Convergence Studies

Convergence study has been carried out for the non-dimensional frequency parameter for isotropic, regular as well as hybrid laminated composite plates. In Fig. 3.1 the rate of convergence of the frequency parameter with the number of terms considered in the series form of the displacement function has been shown, for an antisymmetric 4 layered angle ply laminate. The rate of convergence for the third and the fourth mode have been shown. It

can be seen from the Table 3.2 that for $M = N =, 4$ satisfactory convergence can be achieved even for higher modes upto four. This results in an order of (80 x 80) for the overall stiffness and mass matrices. From Fig. 3.1, it is observed that the nondimensional frequency attains asymptotic values with increase in number of terms.

Table 3.3 gives the percentage change in the values of the non-dimensional frequency parameter with number of terms in the series. It is clear from this table that the change in the value of the frequency parameter has reduced appreciablyfor M and N equal to 4 and that sufficient accuracy has been achieved up to the 4th mode.

3.3 Isotropic Plates

The present formulation developed for the free vibration, studies of hybrid laminated composite plates has been checked by comparing the results obtained with those available in the literature. In Table 3.4 a comparison of the non-dimensional frequency paramater K_n for a simply supported isotropic square plate with length to thickness ratio $(a/h) = 10$, has been presented with the corresponding values obtained by other investigators. Since the length to thickness ratio is 10, this is a thick plate and hence transverse shear deformation effects

can not be neglected. Also for higher modes of vibration rotatory inertia effects can be ignored. This can be clearly seen by comparing the results obtained with those obtained from classical plate theory. The percentage error in the classical plate theory increases for higher modes. It can also be observed that the results obtained using the present solution are in good agreement with those obtained from other methods of solution. The percentage error when compared with the exact solutions of 3-D linear elasticity[32], for the fifth mode is found to be less than 1.2. When compared to those obtained from the higher order theory of Kamal and Durvasula[27], it can be seen that there is not much difference between the two. This is quite expected since the displacement field in both formulations are almost identical. It is observed that with as few number of terms as M = 4 and N = 4 in the series form of displacement function convergence is faster than other methods wherein more number of terms were considered. This is due to the reason that unlike other solution methods here, the stress conditions considered resemble closely that of three dimensional solution method.

It is well known that transverse shear effects reduce the natural frequencies. To see the extent of transverse shear effects on isotropic plates the non-dimensional frequency parameter for square isotropic plate

has been plotted in Fig. 3.2 for length to thickness ratios varying from 10 to 100. It can be seen that for ratios ranging from 100 to 30, the transverse shear effects are almost negligible since they belong to the class of thin plates. Where as after this transverse shear effects become predominant. The frequency parameter reduces by 0.6 percent as the ratio is reduced from 30 to 20 while for 20 to 10 it reduces by 1.86 percent. From the figure it becomes clear that transverse shear effects can not be neglected for plates with length to thickness ratio ≤ 30.

The effect of one more important parameter, that is the aspect ratio of the plate also has been studied and presented in Fig. 3.3. Non-dimensional fundamental frequency parameter for a thin isotropic plate (a/h = 50) for aspect ratios (a/b) varying from one to five has been obtained. It can be seen that the frequency parameter increases, continuously with the aspect ratio. This is because, as the aspect ratio increases the stiffness of the plate reduces hence the natural frequency goes down. Since the non-dimensional parameter has been represented as

$$K_n = \omega_n \, a^2 \sqrt{\frac{\rho}{E_T \, h^2}} \qquad (3.1)$$

the frequency parameter goes up. But this does not mean that the frequency goes up. Table 3.5 shows the corresponding

frequency parameter values when they are defined as below:

$$\bar{K}_n = \omega_n b^2 \sqrt{\frac{\rho}{E_T h^2}}$$
$$\bar{K}_n = K_n / \lambda^2 \qquad (3.2)$$

where

λ - aspect ratio of the plate.

Through out this work we have used the non-dimensional parameter as defined by equation (3.1).

3.4 Composite Laminated Plates - Comparative Study

Table 3.6 shows the values of the non-dimensional frequency parameter (K_n) along with those obtained by other investigators for simply supported antisymmetric rectangular angle ply laminated plate. Values of the frequency parameter for different modes have been given. It can be seen that the values differ for all the modes and all the values are slightly higher than those obtained by other investigators. This is due to the reason that the boundary conditions we have considered are slightly different from those considered by other investigators as explained in Section 2.7. The particular specifications

for the boundary conditions increase the stiffness of the plate, ultimately resulting in higher non-dimensional frequency parameters. Hence in all these even though it is written as simply supported, boundary conditions at the particular edges are slightly different. In the present work we have considered the boundary conditions as given by equation (2.28a).

3.5. Effect of Length to Thickness Ratio and Aspect Ratio

3.5.1 Symmetric Cross Ply Plates

Figure 3.4 shows the variation of the non-dimensional frequency parameter with length to thickness ratio for the two types of Graphite/epoxy laminates. For the symmetric four layered cross ply plates considered, it was found that the non-dimensional parameters were higher for the Graphite-Epoxy laminate for which the ratio of longitudinal to transverse modulus is higher. This is because for this laminate the effective stiffness of the plate is higher. Here also it was seen that the transverse shear effect was predominant as the length to thickness ratio was reduced below 20.

Figure 3.5 shows the plot of the variation of the frequency parameter for regular laminates made of Graphite/Epoxy plies with higher modulus ratio, Kevlar/Epoxy plies and hybrid laminate obtained by stacking Graphite/Epoxy plies as outer layers and Kevlar/Epoxy plies as inner layers. Here also the same type of behaviour with length

to thickness ratio is maintained and that the hybrid laminate has frequency parameters in between the two regular composites. The frequency parameters for the Kevlar/Epoxy plies is lesser when compared to that of the Graphite/Epoxy plies, while for the hybrid laminate the values are closer to that of the Graphite/Epoxy plies.

The effect of aspect ratio on the two types of Graphite/Epoxy plates has been investigated and shown in Fig. 3.6. The frequency parameter increases with aspect ratio and length to thickness ratio indicating that transverse shear effects should be considered for thick plates. Since the Graphite-Epoxy plies with higher modulus gives higher frequencies, we have considered only this grade of Graphite/Epoxy to obtain the hybrid laminate. For aspect ratios < 1, the frequency parameter remains constant and in this region the hybrid composite and Graphite/Epoxy laminates have almost equal non-dimensional frequency parameter values. For aspect ratios > 1, the frequency parameter increases linearly thus indicating high dependence on aspect ratio. Figure 3.7 shows similar behaviour of the frequency parameter with aspect ratio for various length to thickness ratios for regular and hybrid laminates.

3.5.2 Antisymmetric Crossply Laminates

Figures 3.8 and 3.11 show the variation of the non-dimensional frequency parameter with length to thickness ratio and aspect ratio of the plate for anti-symmetric regular and hybrid composite cross ply plates. In this type of construction plies having fibre orientation 0° and 90° are arranged alternatively. The results are presented for a four ply antisymmetric crossply laminate. Since the elements of coupling matrices are non-zero, their effect is to reduce the frequency parameter values. By comparing the results obtained for symmetric and anti-symmetric cases, it can be seen that the amount of reduction for a square hybrid laminate is 25.39% for length to thickness ratio 10 while it is 29.37% for length to thickness ratio 20. But the behaviour with aspect ratio and length to thickness ratio remains same as in the case of a symmetric laminate. Figure 3.8 indicates that the frequency parameter reaches an asymptotic value for aspect ratios upto 4. For plates with aspect ratios upto 2 it can be seen that the transverse shear effects are almost negligible.

3.5.3 Antisymmetric Angleply Laminates

In this type of laminate construction plies having fibre orientation $+\theta$ and $-\theta$ are arranged alternatively. In this type of laminate construction also, the coupling

matrices B, E and F will not be zero. However they tend to zero as the number of layers is increased. Two types of antisymmetric angle plies, $45^\circ/-45^\circ/45^\circ/-45^\circ$ and $30^\circ/-30^\circ/30^\circ/-30^\circ$ have been considered and the variation of the frequency parameter with length to thickness ratio and aspect ratio have been presented in Figures 3.9, 3.10, 3.12 and 3.13. Even though we observe similar behaviour in these cases also, it can be seen that the frequency parameter has highest values for 45° antisymmetric ply due to the fact that the stiffness of a laminate is highest for fibre orientation equal to 45°. It can be seen that the frequency parameters are 6.44 percent higher for a hybrid laminate with length to thickness ratio 10 and aspect ratio 1 than a corresponding 30° antisymmetric laminate. In all these cases it can be seen that the hybrid laminates have intermediate values to those of the regular composites made of Graphite/Epoxy and Kevlar/Epoxy plies. For thicker plates it can be seen that the values for the hybrid laminates are very much near to those of the Graphite/Epoxy plies. It can also be observed that the transverse shear effects for plates with same length to thickness ratio increases with aspect ratio. In the case of antisymmetric angle plies, it can be seen from Fig. 3.9 that the non-dimensional frequency parameter reaches an asymptotic value for aspect ratios upto 3 unlike antisymmetric crossply laminates where it went up to aspect ratios equal to 4.

3.6 Effect of Number of Layers

The effect of number of layers on the non-dimensional parameter has been presented in Fig. 3.14 for symmetric crossply laminated plate and for a angle ply laminnate in Fig. 3.15. It can be seen that for constant length to thickness ratio and aspect ratio the frequency parameter value increases with the number of layers. But after 12 layers it can be seen that the value becomes almost constant. The reason for this is that the plate stiffness coefficients A16, A26, D16, D26, H16 and H26 become negligible with increase in the number of layers since, A16 and A26 are inversely proportional to the number of layers, D16 and D26 are inversely proportional to the third power of the number of layers, H16 and H26 are inversely proportional to the seventh power of the number of layers. The non-dimensional frequency parameter value goes up by 22.18% for hybrid cross ply laminate as the number of layers is increased from 4 to 8, while for 45° symmetric angle ply laminates this value is 24.1%.

3.7 Effect of Fibre Orientation

Figure 3.16 shows the variation of the fundamental non-dimensional frequency parameter with fibre orientation for square regular as well as hybrid composite laminated plates. From the figure it can be seen that the variation

of non-dimensional parameter is almost a normal distribution curve with maximum value at 45° fibre orientation since the plate stifness will be maximum for this fibre orientation.

Table 3.1

Composite Lamina Properties

Properties	Composite Materials		
	Material-I[22]	Material-II[22]	Material-III[10]
E_L/E_T	40.0	25.0	14.82
G_{LT}/E_T	0.6	0.5	0.375
G_{TZ}/E_T	0.5	0.2	0.375
G_{LZ}/E_T	0.5	0.2	0.375
ν_{LT}	0.25	0.25	0.34

Material I - Graphite/Epoxy (High Modulus variety)

Material II - Graphite/Epoxy (Low Modulus variety)

Material III - Kevlar/Epoxy

Table 3.2

Convergence of the Non-dimensional Frequency Parameter with Number of Terms for Antisymmetric angle Ply Laminate +

M	N	3rd Mode	4th Mode	5th Mode	6th Mode	7th Mode
1	2	46.5103	-	-	-	-
2	2	44.3101	47.6528	69.1678	-	-
2	3	43.5417	47.0058	68.2643		
3	3	42.8901	45.6913	62.3848*	62.6739*	74.1432
3	4	42.5878	45.3915	62.0606*	62.4431*	71.4482
4	4	42.3430	45.1760	60.1573*	61.4249*	70.3716

* Degenerate Modes.

Tabl 3.3

Percentage Change in the Values of the Non-Dimensional Frequency Parameter with Number of Terms for Antisymmetric Angleply Laminate +

M	N	Percentage Change 3rd Mode	4th Mode	5th Mode	6th Mode	7th Mode
1	2					
		4.73				
2	2					
		1.73	1.3	1.3		
2	3					
		1.49	2.79	8.61		
3	3					
		0.70	0.656	0.52	0.37	3.63
3	4					
		0.57	0.47	3.07	1.63	1.51
4	4					

+ $E_L/E_T = 40$; $G_{LT}/E_T = 0.6$; $G_{LZ}/E_T = 0.5$; $G_{TZ}/E_T = 0.5$; $\nu_{LT}=0.25$

$a/b = 1$; $+45°/-45°/+45°/-45$, $a/h = 10$.

Table 3.4

Comparison of Non-dimensional Frequencies $K_n = \omega_n a^2 \sqrt{\frac{\rho}{E h^2}}$ for a Square Isotropic Plate$

Mode Designation M	Mode Designation N	Classical plate theory	FEM solution including shear deformation[22] 3 dof (4x2)	FEM solution including shear deformation[22] 5dof (2x2)	Mindlin's thick plate solution[33]	3-D linear elasticity solution[32]	Rock and Hinton[34]	Present solution	Ka an Du su
1	1	5.973	5.793	5.920	5.767	5.780	5.774	5.78861	5
1	2	14.934	14.081	15.251	13.755	13.805	13.749	13.8682	14
1	3	29.867	27.545	28.133	25.700	25.867	27.207	26.0729	25
2	3	38.829	35.050	33.057	32.230	32.491	34.010	32.7966	32
3	3	53.868	49.693	49.723	42.302	42.724	48.609	43.2134	43

$ ν = 0.3, a/h = 10.

Table 3.5

Comparison between the Two Types of Non-dimensional Frequency Parameters

Aspect ratio (a/b)	$K_n = \omega_n a^2 (\rho / E_T h^2)^{1/2}$	$\bar{K}_n = \omega_n b^2 (\rho / E_T h^2)^{1/2}$
1	5.96429	5.96429
2	14.9164	3.7291
3	29.8739	3.3193
4	50.7440	3.1715
5	77.6303	3.1052

Table 3.6

Comparison of Non-dimensional Frequency Parameter, $K_n = \omega_n a^2 \sqrt{\frac{\rho}{E_T h^2}}$ of a Four Layered Simply Supported Composite Plate+

ignation N	Classical plate theory	FEM (Based on YNS Theory)[22] Half plate(2x2) ndf=5	Half plate(4x2) ndf=3	Kamal and Durvasula[27]	Present solution (4x4)	Bert and
1	25.53	18.259	19.153	19.006	23.6899	18.46
2	53.74	35.585	35.405	35.521	42.3430	34.87
2	94.11	-	-	51.560	70.3716	50.52
3	98.87	54.367	55.390	56.540	60.1573	54.27
3	147.65	70.315	67.637	75.291	74.3250	67.17
3	211.75	99.597	84.725	93.746	84.3284	82.84

Table 3.3 for material properties.

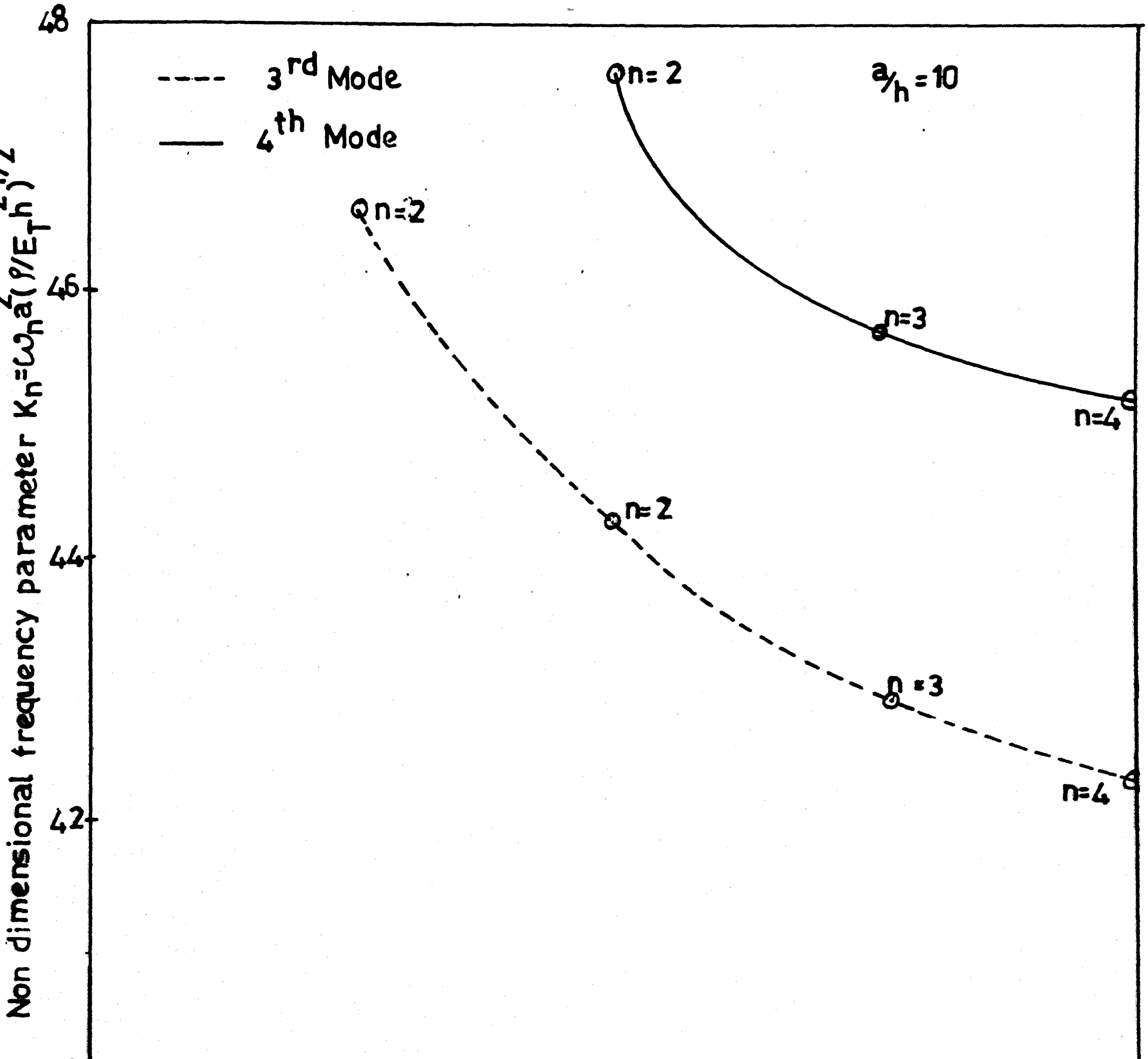

Fig: 3·1 Convergence study; Square antisymmetric Graphite Epoxy laminate (45°/-45°/45°/-45°)

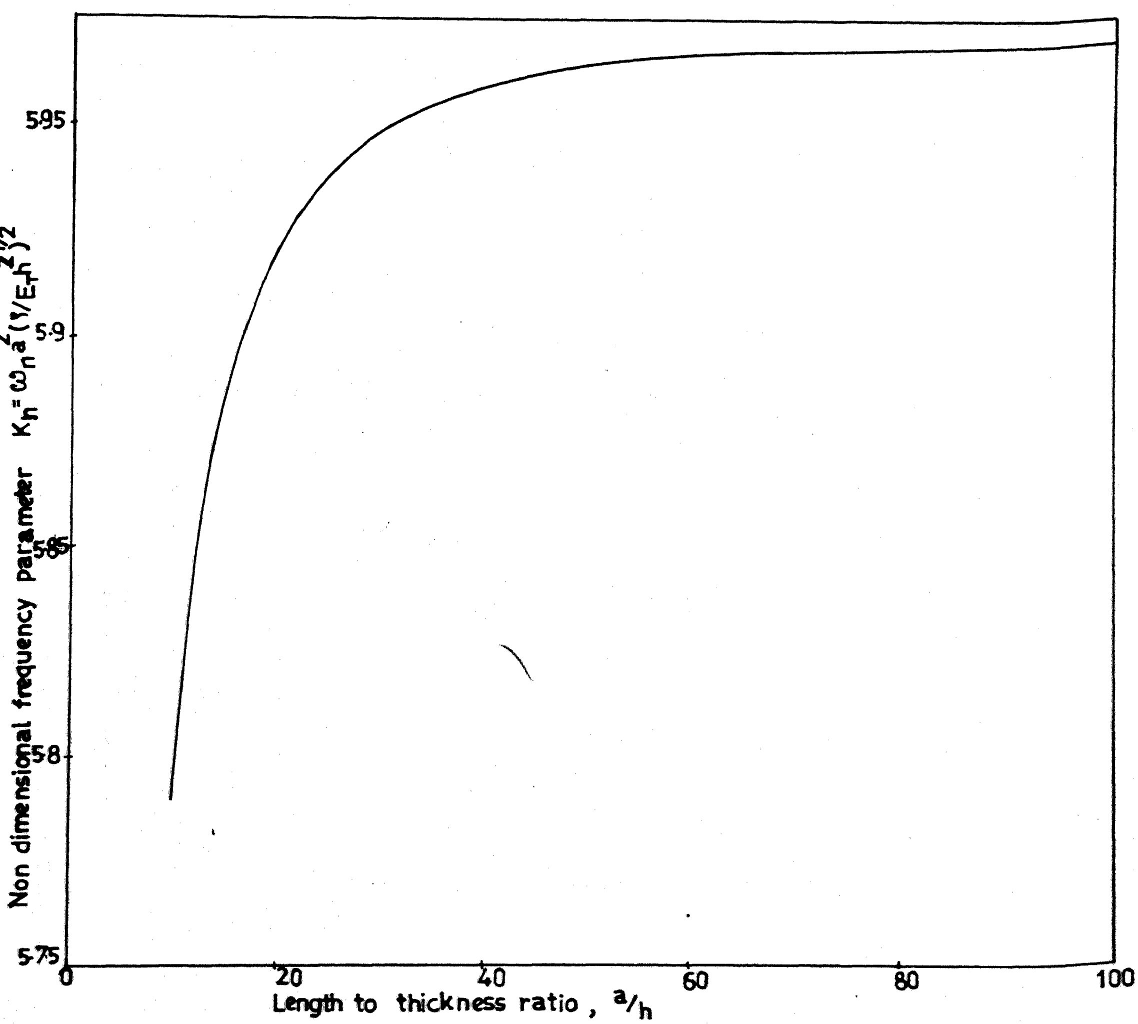

Fig: 3.2 Variation of non dimensional frequency parameter with length to thickness ratio for an isotropic square plate.

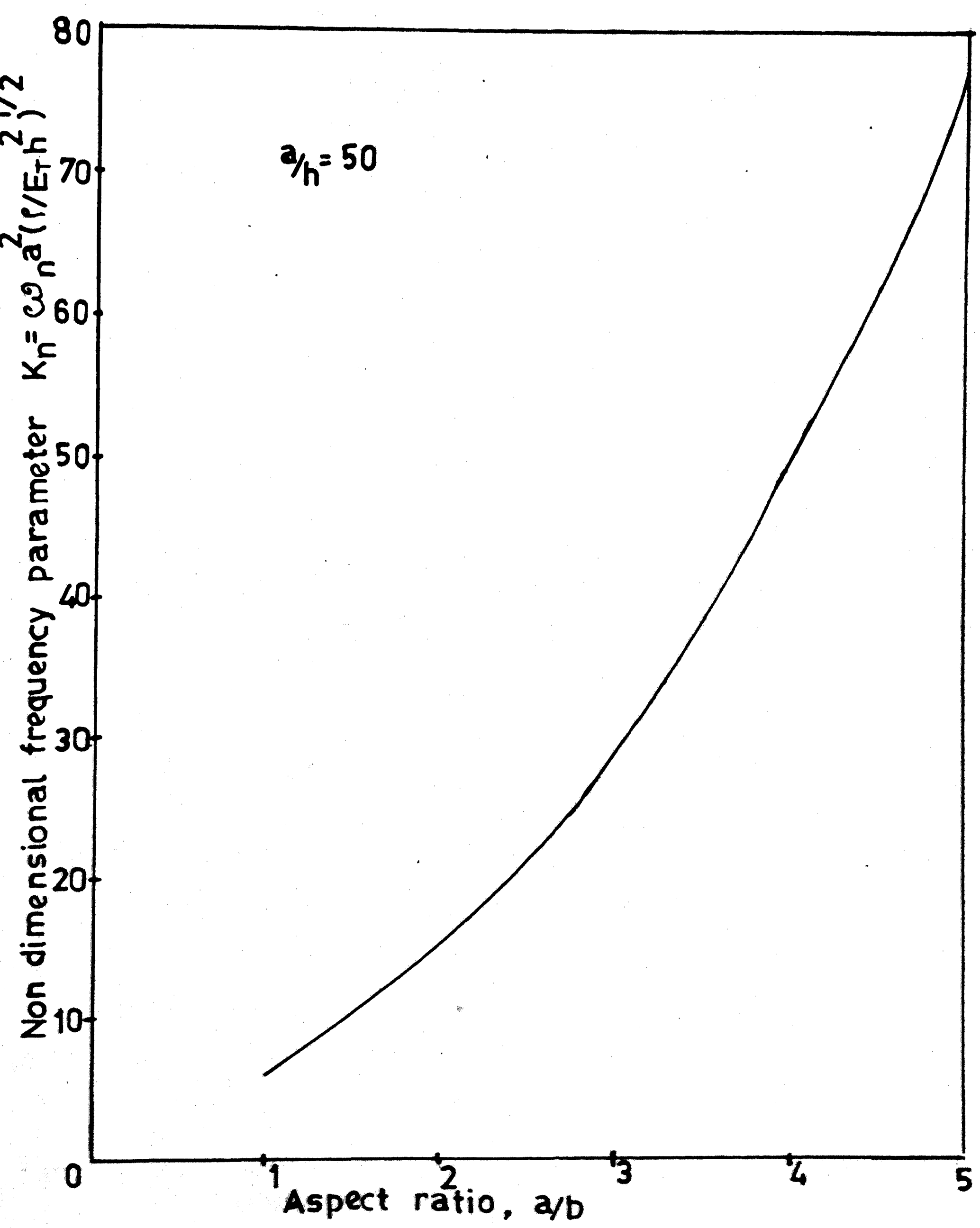

Fig: 3·3 Variation of non dimensional frequency parameter with aspect ratio for an isotropic plate

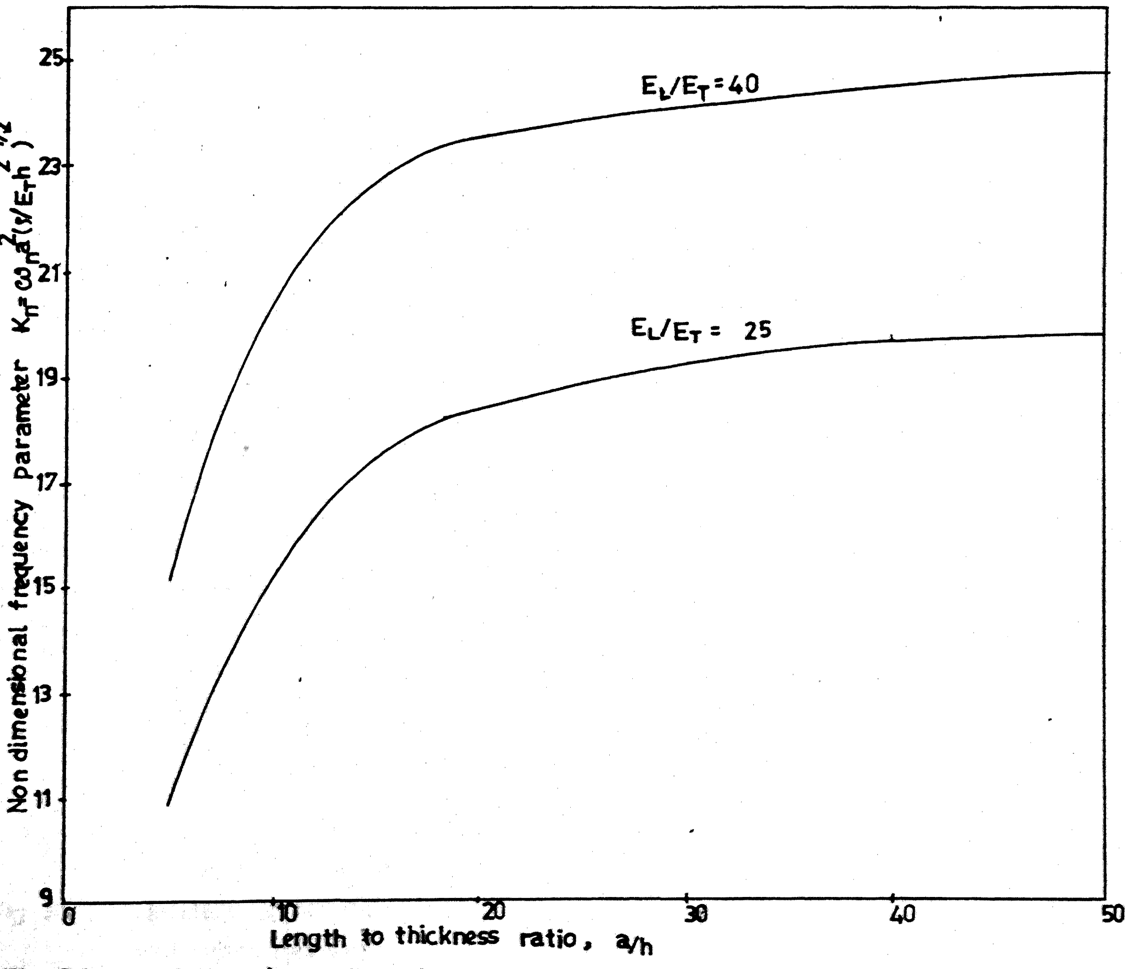

Fig. 3.4 Variation of non dimensional frequency parameter with length to thickness ratio for symmetric cross ply square plates

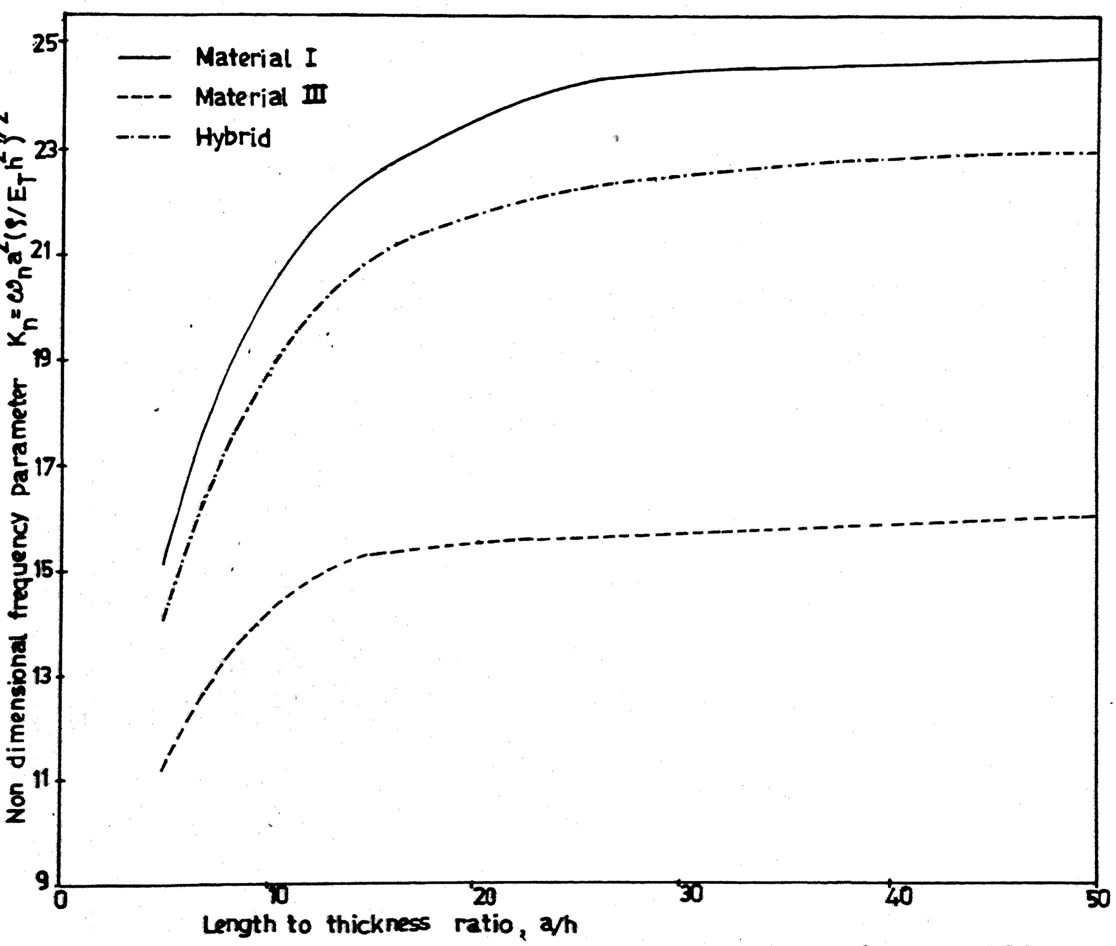

Fig: 3·5 Variation of non dimensional frequency parameter with length to thickness ratio for symmetric cross ply square plates.

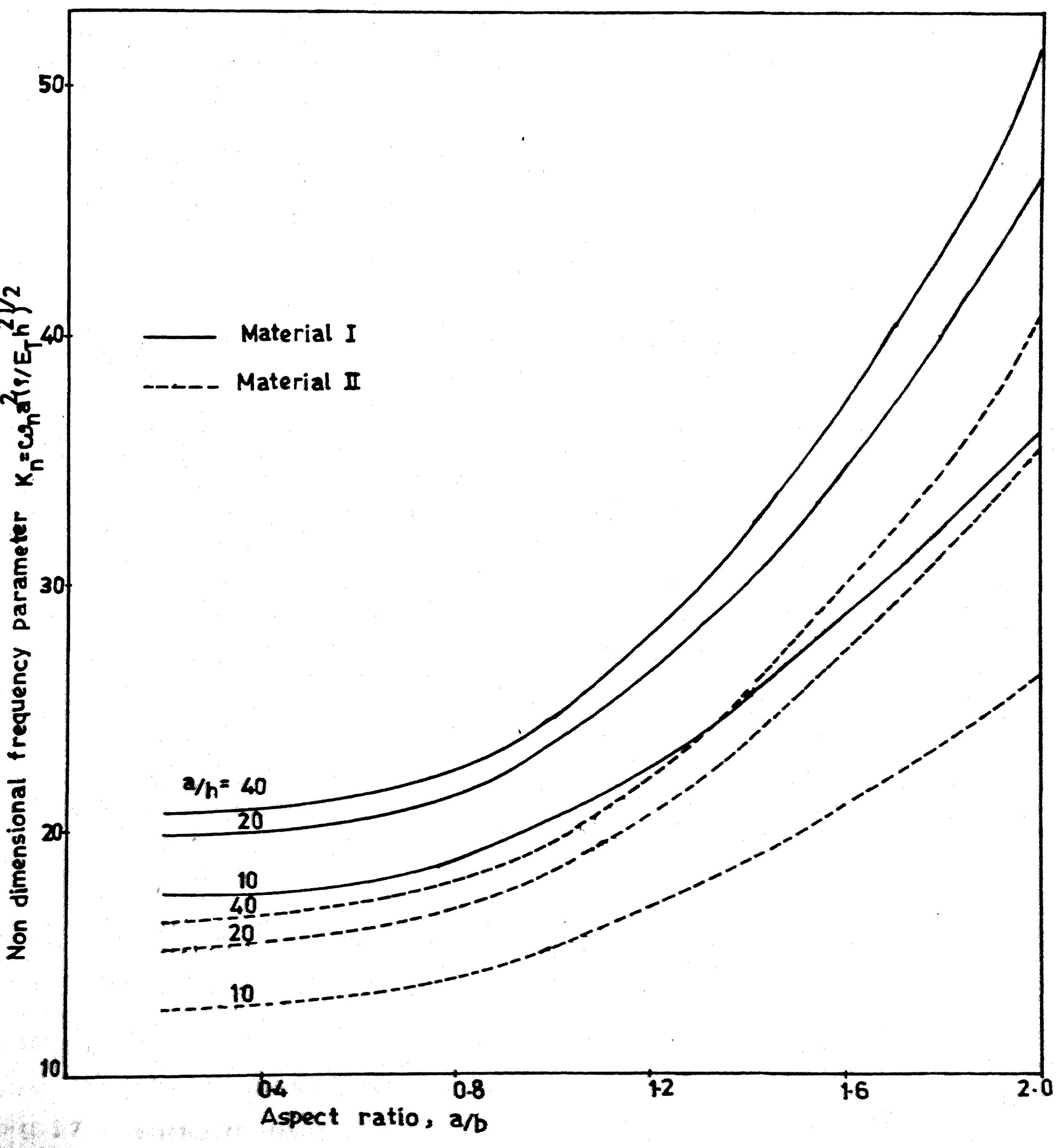

Fig: 3·6 Variation of non dimensional frequency parameter with aspect ratio for symmetric cross ply plates.

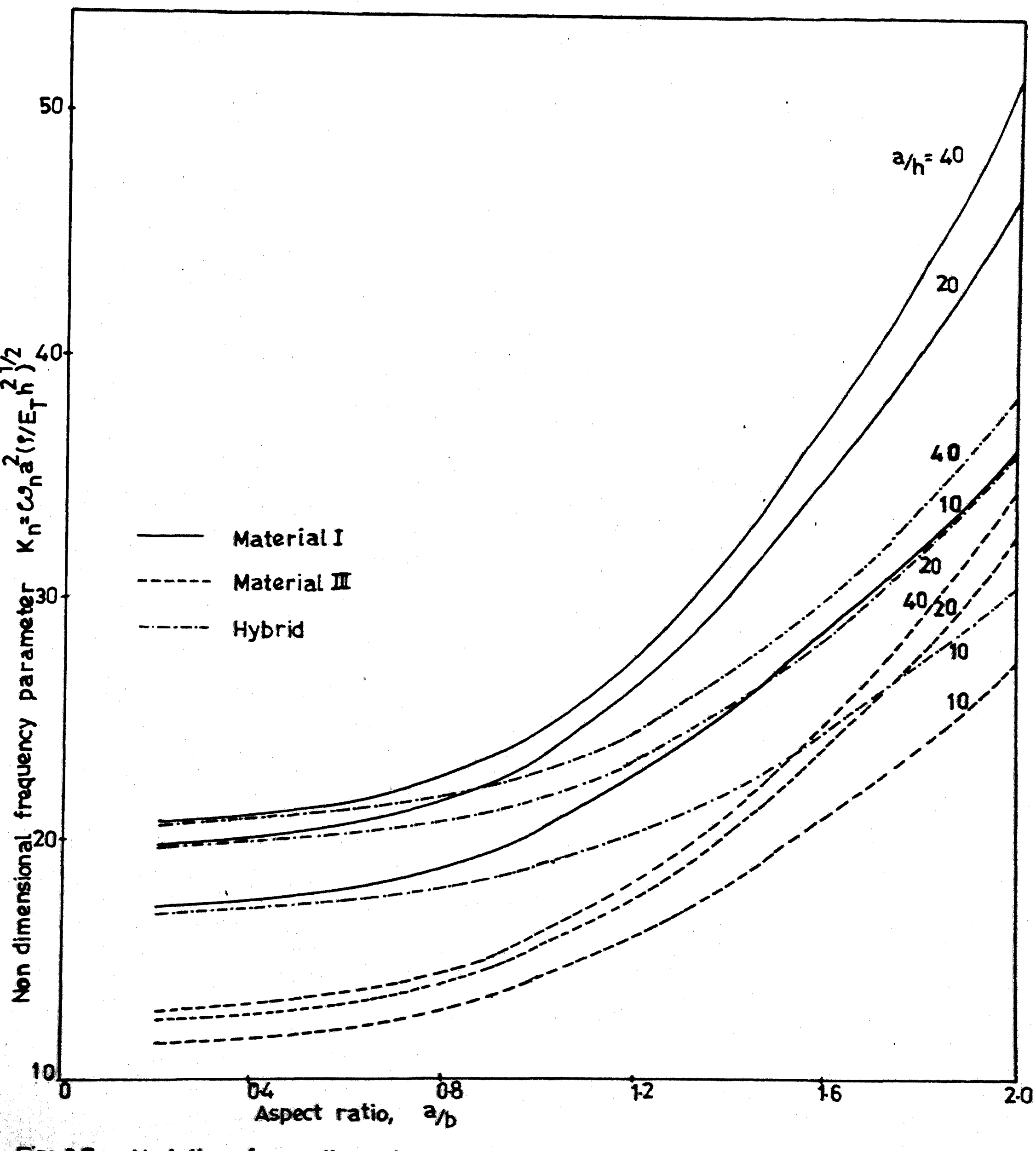

Fig: 3.7 Variation of non dimensional frequency parameter with aspect ratio for symmetric cross ply plates.

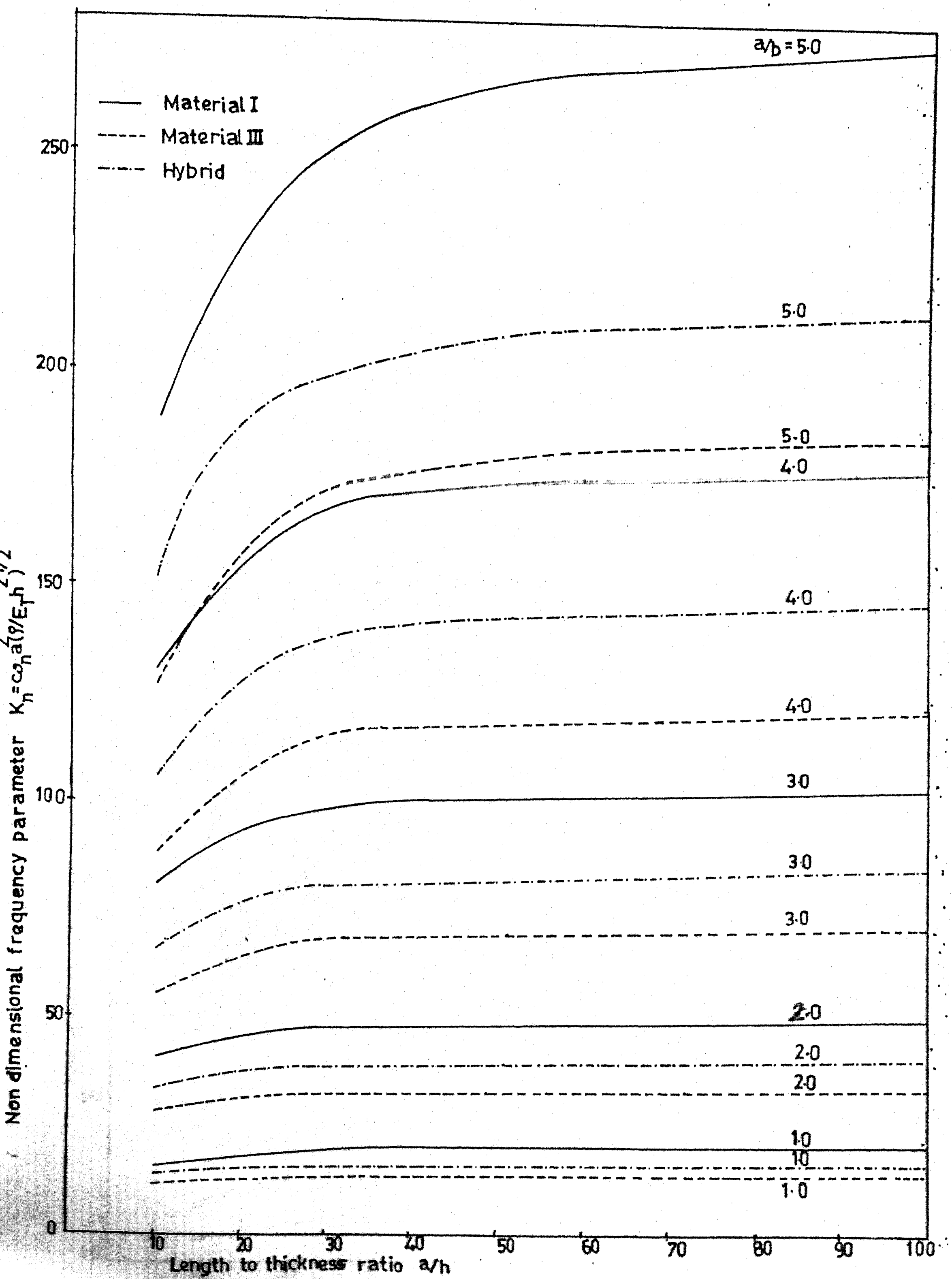

Fig: 3.8 Variation of non dimensional frequency parameter with length to thickness ratio for antisymmetric cross ply laminated plates

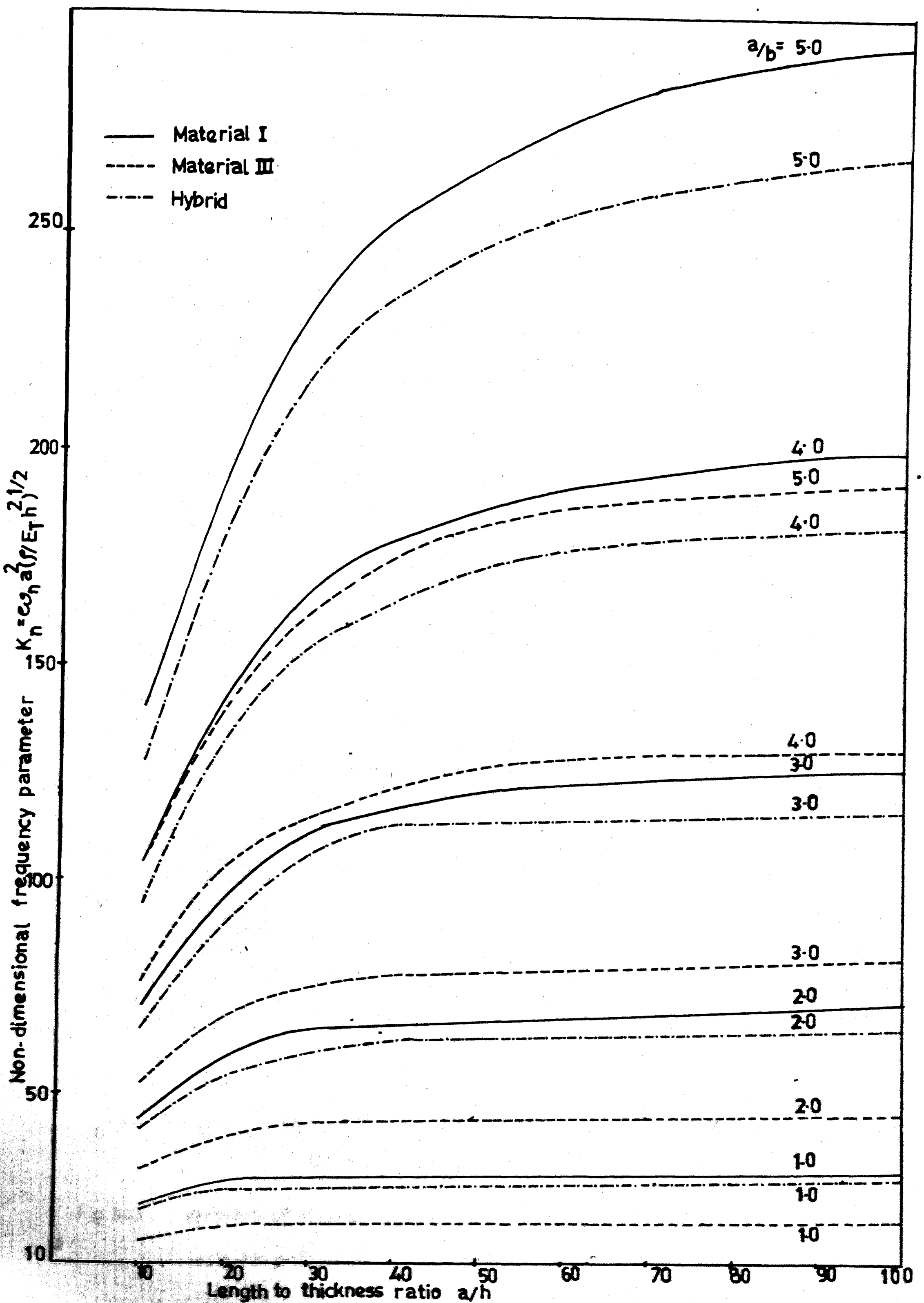

Fig: 3.9 Variation of non-dimensional frequency parameter with length to thickness ratio for antisymmetric angle ply laminated plates (45°/-45°/45°/-45°)

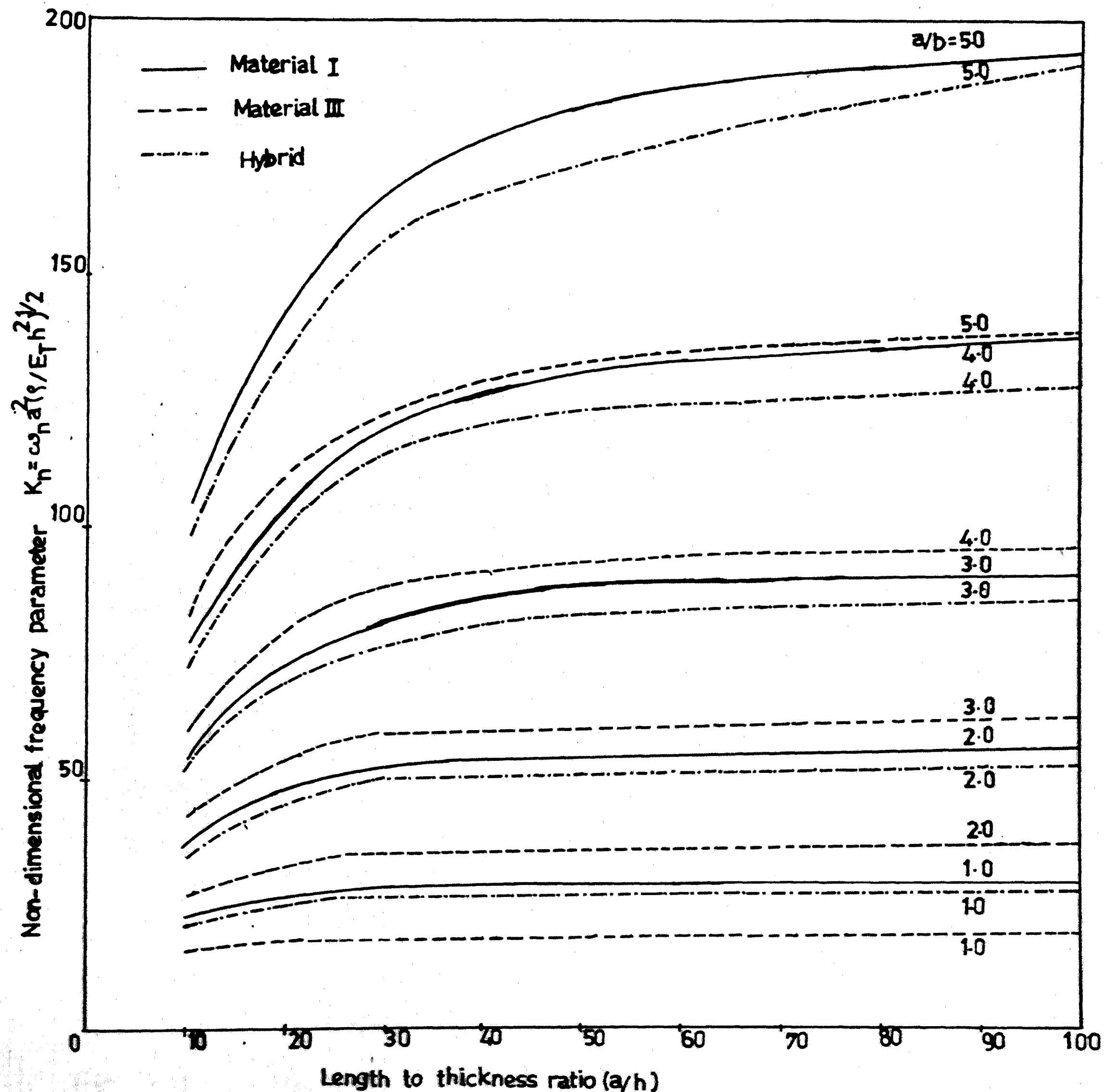

Fig. 3.10 Variation of non-dimensional frequency parameter with length to thickness ratio for antisymmetric angle ply laminated plate (30°/-30°/30°/30°)

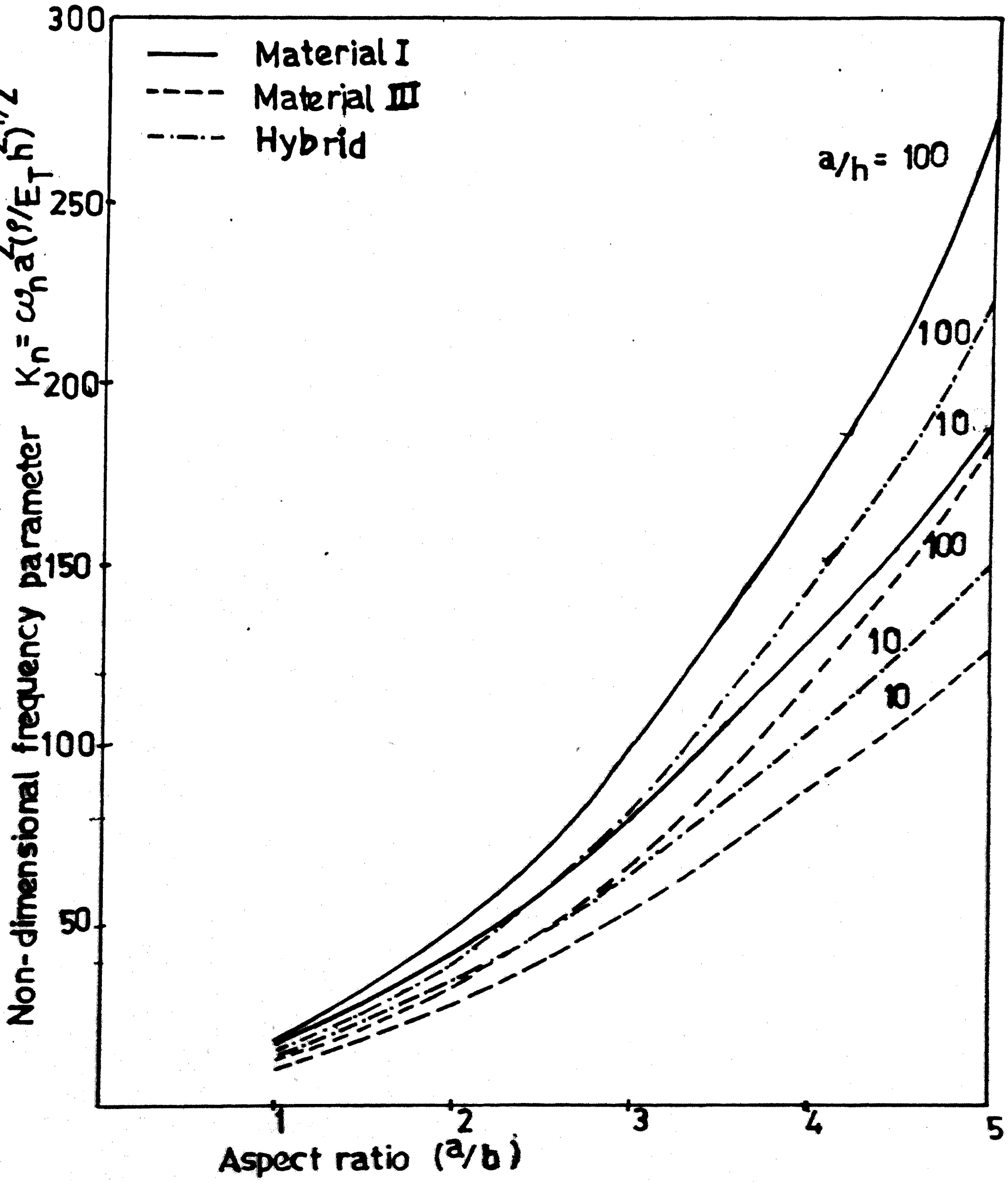

Fig: 3·11 Variation of non-dimensional frequency parameter with aspect ratio for antisymmetric cross ply plates (0°/90°/0°/90°)

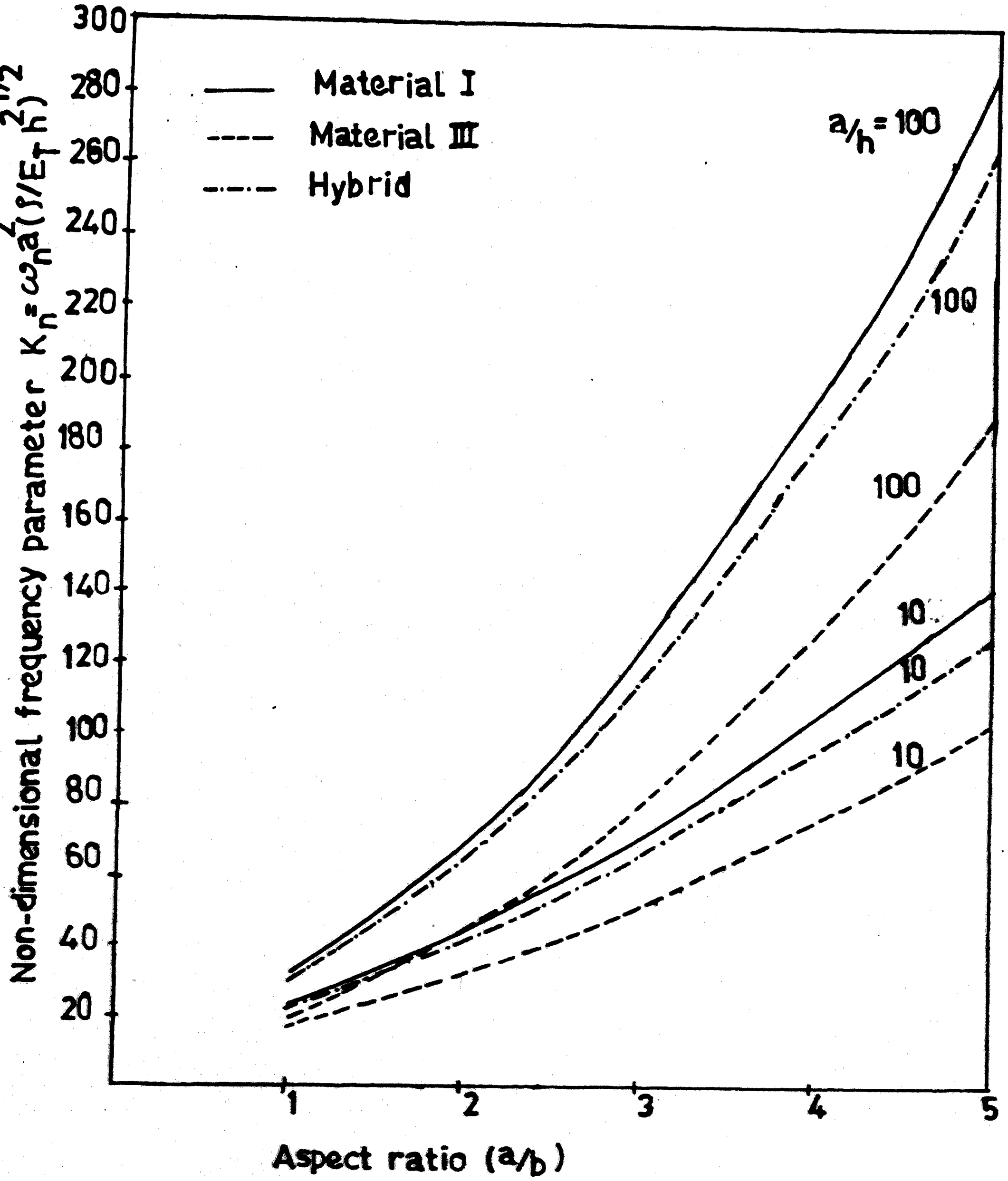

Fig: 3.12 Variation of non-dimensional frequency parameter with aspect ratio for antisymmetric plates (45°/-45°/45°/-45°)

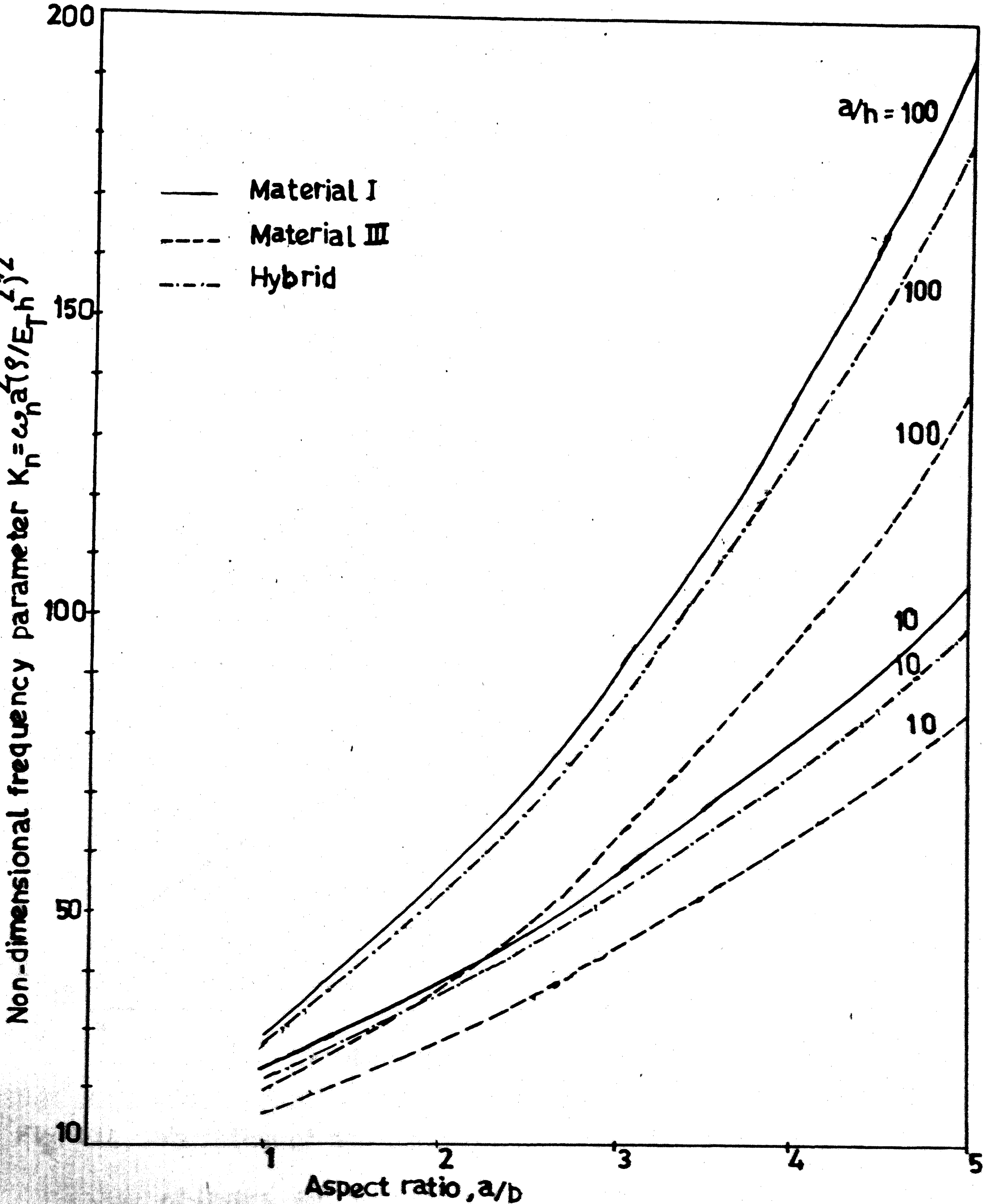

Fig:3·13 Variation of non-dimensional frequency parameter with aspect ratio for antisymmetric laminates (30°/-30°/30°/-30°)

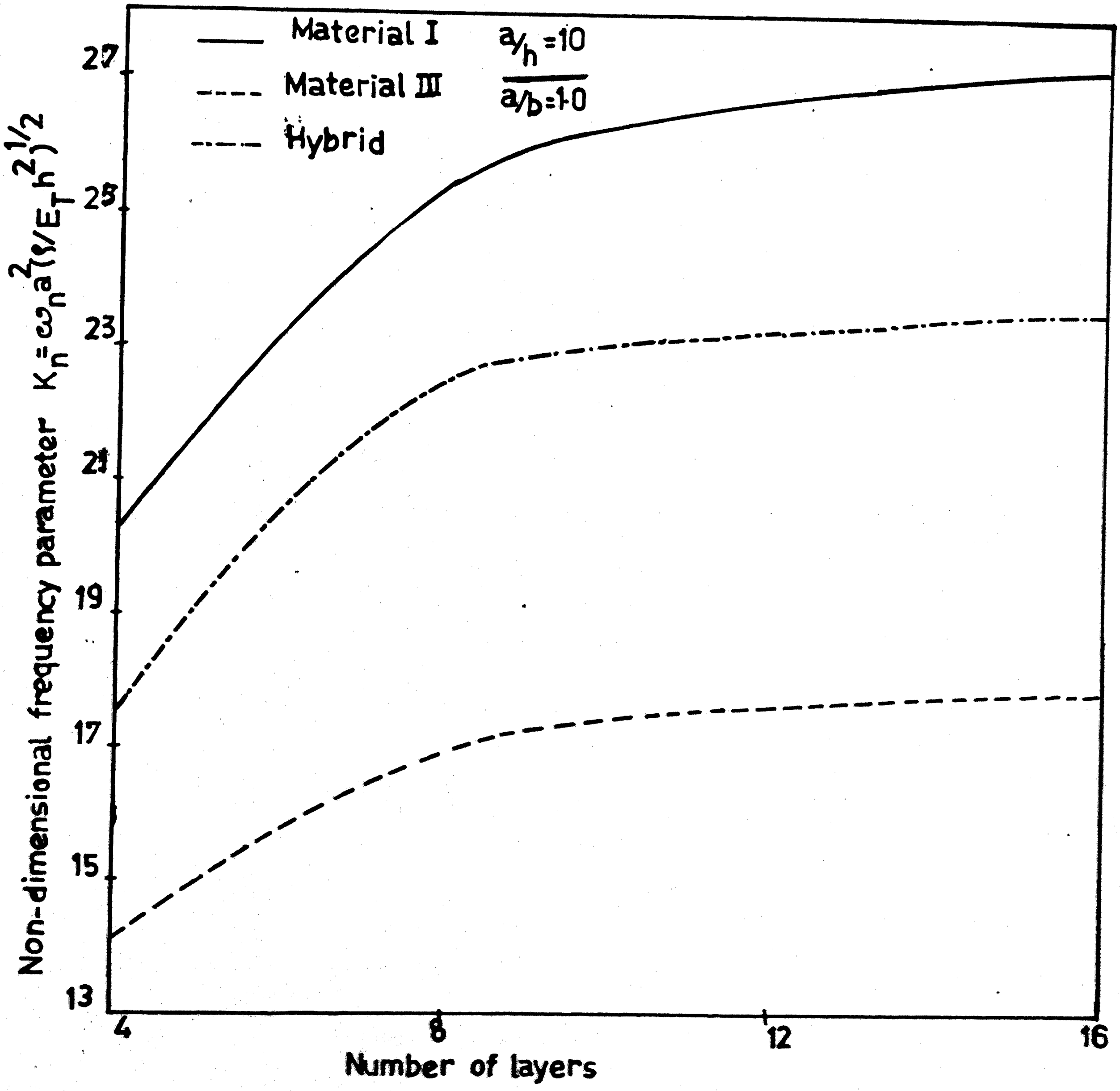

Fig: 3.14 Variation of non-dimensional frequency parameter with number of layers for symmetric cross-ply plates

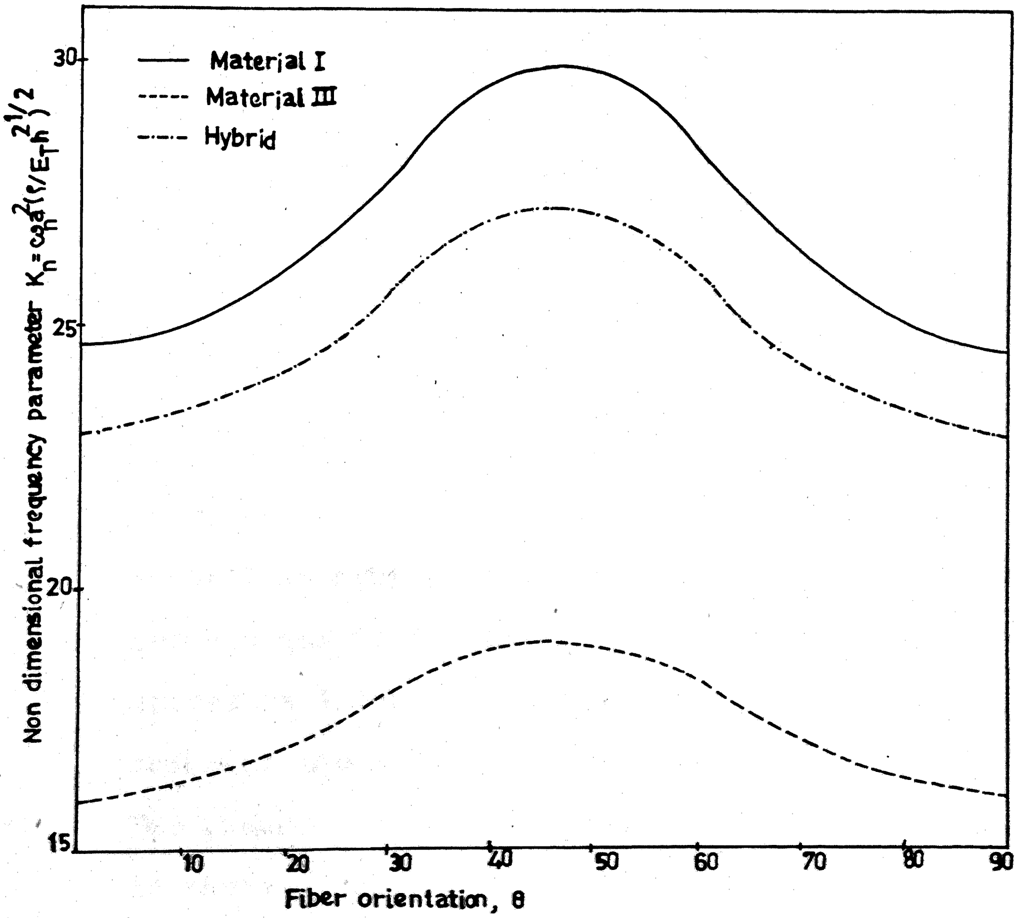

Fig: 3.16 Variation of non dimensional frequency parameter with fiber orientation for symmetric laminates

CHAPTER 4

CONCLUSIONS AND SCOPE FOR FURTHER EXTENSION

Use of hybrid laminated plates is an effective means to achieve intermediate properties and to tailor properties as per design requirement using various parameters such as stacking sequence, number of layers, fibre orientation etc. From the limited studies carried out on the free vibration response of regular as well as hybrid laminated composite plates, it can be concluded that

(1) Rayleigh-Ritz energy method can be successfully employed for the free vibration analysis of regular as well as hybrid composite plates. Appreciable convergence of the solution can be achieved with as few as 4 terms in the summation series thus reducing the computational time.

(2) The properties of the hybrid laminate are intermediate to those of single material composites.

(3) The transverse shear effects can not be neglected for plates with length to thickness ratio less than or equal to 20 and rotatory inertia effects can not be neglected for higher modes.

(4) The natural frequency of vibration reduces as the plate aspect ratio increases.

(5) For laminates with any stacking sequence fibre orientations $\pm 45^\circ$ give the highest fundamental frequency.

Scope for further extension

The following is the list of several possible interesting aspects for extending the present work

(1) For free vibration analysis different boundary conditions can be considered. It is very important to investigate the free vibration response of plates with mixed boundary conditions.

(2) Transverse shear and rotatory inertia effects on trapezoidal as well as skew plates can be studied.

(3) The effect of delamination on natural frequencies can be investigated.

(4) The present formulation can be used to study the buckling and bending response of hybrid composite plates under various boundary and loading conditions.

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APPENDIX A

The list of the integrals involved in this formulation are as follows:

$$I1_{mr}^{pq} = \int_0^1 \phi_{um}^{p}(\xi)\, \phi_{ur}^{q}(\xi)\, d\xi; \quad J1_{ns}^{pq} = \int_0^1 \Psi_{un}^{p}(\eta) \cdot \Psi_{us}^{q}(\eta)\, d\eta$$

$$I2_{mr}^{pq} = \int_0^1 \phi_{vm}^{p}(\xi)\, \phi_{vr}^{q}(\xi)\, d\xi; \quad J2_{ns}^{pq} = \int_0^1 \Psi_{vn}^{p}(\eta) \cdot \Psi_{vs}^{q}(\eta) \cdot d\eta$$

$$I3_{mr}^{pq} = \int_0^1 \phi_{wm}^{p}(\xi)\, \phi_{wr}^{q}(\xi)\, d\xi; \quad J3_{ns}^{pq} = \int_0^1 \Psi_{wn}^{p}(\eta) \cdot \Psi_{ws}^{q}(\eta) \cdot d\eta$$

$$I4_{mr}^{pq} = \int_0^1 \phi_{xm}^{p}(\xi) \cdot \phi_{xr}^{q}(\xi)\, d\xi; \quad J4_{ns}^{pq} = \int_0^1 \Psi_{xn}^{p}(\eta) \cdot \Psi_{xs}^{q}(\eta) \cdot d\eta$$

$$I5_{mr}^{pq} = \int_0^1 \phi_{ym}^{p}(\xi) \cdot \phi_{yr}^{q}(\xi)\, d\xi; \quad J5_{ns}^{pq} = \int_0^1 \Psi_{yn}^{p}(\eta) \cdot \Psi_{ys}^{q}(\eta) \cdot d\eta$$

$$I6_{mr}^{pq} = \int_0^1 \phi_{um}^{p}(\xi) \cdot \phi_{vr}^{q}(\xi)\, d\xi; \quad J6_{ns}^{pq} = \int_0^1 \Psi_{un}^{p}(\eta) \cdot \Psi_{vs}^{q}(\eta) \cdot d\eta$$

$$I7_{mr}^{pq} = \int_0^1 \phi_{um}^{p}(\xi) \cdot \phi_{wr}^{q}(\xi)\, d\xi; \quad J7_{ns}^{pq} = \int_0^1 \Psi_{un}^{p}(\eta) \cdot \Psi_{ws}^{q}(\eta)\, d\eta$$

$$I8_{mr}^{pq} = \int_0^1 \phi_{um}^{p}(\xi) \cdot \phi_{xr}^{q}(\xi)\, d\xi; \quad J8_{ns}^{pq} = \int_0^1 \Psi_{un}^{p}(\eta) \cdot \Psi_{xs}^{q}(\eta) \cdot d\eta$$

$$I9_{mr}^{pq} = \int_0^1 \phi_{um}^{p}(\xi) \cdot \phi_{yr}^{q}(\xi)\, d\xi; \quad J9_{ns}^{pq} = \int_0^1 \Psi_{un}^{p}(\eta) \cdot \Psi_{ys}^{q}(\eta) \cdot d\eta$$

$$I10_{mr}^{pq} = \int_0^1 \phi_{vm}^p(\xi) \cdot \phi_{wr}^q(\xi)\, d\xi; \quad J10_{ns}^{pq} = \int_0^1 \Psi_{vn}^p(\eta) \cdot \Psi_{ws}^q(\eta) \cdot d\eta$$

$$I11_{mr}^{pq} = \int_0^1 \phi_{vm}^p(\xi) \cdot \phi_{xr}^q(\xi)\, d\xi; \quad J11_{ns}^{pq} = \int_0^1 \Psi_{vn}^p(\eta) \cdot \Psi_{xs}^q(\eta) \cdot d\eta$$

$$I12_{mr}^{pq} = \int_0^1 \phi_{vm}^p(\xi) \cdot \phi_{yr}^q(\xi)\, d\xi; \quad J12_{ns}^{pq} = \int_0^1 \Psi_{vn}^p(\eta) \cdot \Psi_{ys}^q(\eta) \cdot d\eta$$

$$I13_{mr}^{pq} = \int_0^1 \phi_{xm}^p(\xi) \cdot \phi_{wr}^q(\xi)\, d\xi; \quad J13_{ns}^{pq} = \int_0^1 \Psi_{xn}^p(\eta) \cdot \Psi_{ws}^q(\eta) \cdot d\eta$$

$$I14_{mr}^{pq} = \int_0^1 \phi_{ym}^p(\xi) \cdot \phi_{wr}^q(\xi)\, d\xi; \quad J14_{ns}^{pq} = \int_0^1 \Psi_{yn}^p(\eta) \cdot \Psi_{ws}^q(\eta) \cdot d\eta$$

$$I15_{mr}^{pq} = \int_0^1 \phi_{xm}^p(\xi) \cdot \phi_{yr}^q(\xi)\, d\xi; \quad J15_{ns}^{pq} = \int_0^1 \Psi_{xn}^p(\eta) \cdot \Psi_{ys}^q(\eta) \cdot d\eta$$

For the simply supported conditions specified in Section 2.7 and admissible functions given in Section 2.8, the values of the integrals have been listed. These integrals have been obtained by numerical integration technique using available routines in the DEC-1090 system.

```
                    THE VALUES OF THE INTEGRAL I1mr00 ARE AS FOLLOWS

.500000007450580597D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .500000007450580597D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .500000007450580597D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .500000007450580597D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .499999992549419403D+00
                    THE VALUES OF THE INTEGRAL I1mr01 ARE AS FOLLOWS

.000000000000000000D+00  .133333332245618321D+01  .000000000000000000D+00  .533333375795851822D+00  .000000000000000000D+00
.133333336926956175D+01  .000000000000000000D+00  .240000005064119759D+01  .000000000000000000D+00  .952381028206878253D+00
.000000000000000000D+00 -.240000013256461003D+01  .000000000000000000D+00  .342857102275077293D+01  .000000000000000000D+00
.533333340685817919D+00  .000000000000000000D+00 -.342857096423104976D+01  .000000000000000000D+00  .444444448620957494D+01
.000000000000000000D+00 -.952380957986810448D+00  .000000000000000000D+00 -.444444441598950713D+01  .000000000000000000D+00
                    THE VALUES OF THE INTEGRAL I1mr10 ARE AS FOLLOWS

.000000000000000000D+00 -.133333331075283858D+01  .000000000000000000D+00 -.533333328982473285D+00  .000000000000000000D+00
.133333334586287248D+01  .000000000000000000D+00 -.240000003893785295D+01  .000000000000000000D+00 -.952380981393499716D+00
.000000000000000000D+00  .240000019108133320D+01  .000000000000000000D+00 -.342857131533438878D+01  .000000000000000000D+00
.533333334834145602D+00  .000000000000000000D+00  .342857149088455830D+01  .000000000000000000D+00 -.444444441598950713D+01
.000000000000000000D+00  .952380925802612704D+00  .000000000000000000D+00  .444444454472629811D+01  .000000000000000000D+00
                    THE VALUES OF THE INTEGRAL I1mr11 ARE AS FOLLOWS

.493480220054467931D+01  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .197392090963158494D+02  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .444132191430935663D+02  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .789568163839384081D+02  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .123370051336902830D+03
                    THE VALUES OF THE INTEGRAL I2mr00 ARE AS FOLLOWS

.500000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .500000007450580597D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .499999992549419403D+00  .000000000000000000D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .499999988079071044D+00  .000000000000000000D+00
.000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .000000000000000000D+00  .499999985098838806D+00
                    THE VALUES OF THE INTEGRAL I2mr01 ARE AS FOLLOWS

.000000000000000000D+00  .266666662150567716D+01  .000000000000000000D+00  .213333331592989314D+01  .000000000000000000D+00
.666666672931436241D+00  .000000000000000000D+00  .360000005840677943D+01  .000000000000000000D+00  .238095245348374929D+01
.000000000000000000D+00 -.160000012738755546D+01  .000000000000000000D+00  .457142842044585171D+01  .000000000000000000D+00
.133333337085364010D+00  .000000000000000000D+00 -.257142861816341872D+01  .000000000000000000D+00  .555555551998688391D+01
.000000000000000000D+00 -.380952370321045082D+00  .000000000000000000D+00 -.355555563578103849D+01  .000000000000000000D+00
                    THE VALUES OF THE INTEGRAL I2mr10 ARE AS FOLLOWS

.000000000000000000D+00 -.666666661228091606D+00  .000000000000000000D+00 -.133333334394896295D+00  .000000000000000000D+00
.266666673853912350D+01  .000000000000000000D+00 -.160000003376079839D+01  .000000000000000000D+00 -.380952411282751301D+00
.000000000000000000D+00  .360000019884691504D+01  .000000000000000000D+00 -.257142826706307970D+01  .000000000000000000D+00
.213333336274327168D+01  .000000000000000000D+00  .457142795231206635D+01  .000000000000000000D+00 -.355555558896765995D+01
.000000000000000000D+00  .238095239496702612D+01  .000000000000000000D+00  .555555551998688391D+01  .000000000000000000D+00
```

THE VALUES OF THE INTEGRAL I2mr11 ARE AS FOLLOWS

.493480227407896236D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132204667106612D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568363852633977D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370053175259906D+03

THE VALUES OF THE INTEGRAL I3mr00 ARE AS FOLLOWS

.500000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.499999992549419403D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999880790710449D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999985098838806D+00

THE VALUES OF THE INTEGRAL I3mr01 ARE AS FOLLOWS

.000000000000000000D+00	.266666662150567716D+01	.000000000000000000D+00	.213333331592989314D+01	.000000000000000000D+00
-.666666672931436241D+00	.000000000000000000D+00	.360000005840677943D+01	.000000000000000000D+00	.238095245348374929D+01
.000000000000000000D+00	-.160000012738755546D+01	.000000000000000000D+00	.457142842044585171D+01	.000000000000000000D+00
-.133333333708536101D+00	.000000000000000000D+00	-.257142861816341872D+01	.000000000000000000D+00	.555555551998688391D+01
.000000000000000000D+00	-.380952370321045082D+00	.000000000000000000D+00	-.355555563578103849D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I3mr10 ARE AS FOLLOWS

.000000000000000000D+00	-.666666661228091606D+00	.000000000000000000D+00	-.133333343948962955D+00	.000000000000000000D+00
.266666673853912350D+01	.000000000000000000D+00	-.160000003376079839D+01	.000000000000000000D+00	-.380952411282751301D+00
.000000000000000000D+00	.360000019884691504D+01	.000000000000000000D+00	-.257142826706307970D+01	.000000000000000000D+00
.213333336274327168D+01	.000000000000000000D+00	.457142795231206635D+01	.000000000000000000D+00	-.355555558896765995D+01
.000000000000000000D+00	.238095239496702612D+01	.000000000000000000D+00	.555555551998688391D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I3mr02 ARE AS FOLLOWS

-.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.123370051336902830D+03

THE VALUES OF THE INTEGRAL I3mr20 ARE AS FOLLOWS

-.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.123370051336902830D+03

THE VALUES OF THE INTEGRAL I3mr11 ARE AS FOLLOWS

.493480227407896236D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132204667106612D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568363852633977D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370053175259906D+03

THE VALUES OF THE INTEGRAL I3mr12 ARE AS FOLLOWS

.000000000000000000D+00	.263189448548652856D+02	.000000000000000000D+00	.210551577320103358D+02	.000000000000000000D+00
-.263189457789243392D+02	.000000000000000000D+00	.142122306374538284D+03	.000000000000000000D+00	.939962398750461419D+02
.000000000000000000D+00	-.142122311225818315D+03	.000000000000000000D+00	.406063675867061664D+03	.000000000000000000D+00
-.210551563459217553D+02	.000000000000000000D+00	-.406063668936618761D+03	.000000000000000000D+00	.877298177229827080D+03
.000000000000000000D+00	-.939962329446032394D+02	.000000000000000000D+00	-.877298163368941275D+03	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I3mr21 ARE AS FOLLOWS

.000000000000000000D+00	-.263189446236505222D+02	.000000000000000000D+00	-.210551558838922285D+02	.000000000000000000D+00
.263189453168948124D+02	.000000000000000000D+00	-.142122305681493993D+03	.000000000000000000D+00	-.939962352547508736D+02
.000000000000000000D+00	.142122314691069767D+03	.000000000000000000D+00	-.406063710519276177D+03	.000000000000000000D+00
.210551561149069919D+02	.000000000000000000D+00	.406063731310604884D+03	.000000000000000000D+00	-.877298163368941275D+03
.000000000000000000D+00	.939962297681502424D+02	.000000000000000000D+00	.877298188780565251D+03	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I3mr22 ARE AS FOLLOWS

.487045455170012186D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.779272739884088036D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.394506812809100173D+04	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.124683606796627662D+05	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.304403400409329071D+05

THE VALUES OF THE INTEGRAL I4mr00 ARE AS FOLLOWS

.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00

.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999992549419403D+00

THE VALUES OF THE INTEGRAL I4mr01 ARE AS FOLLOWS

.000000000000000000D+00	.133333332215618321D+01	.000000000000000000D+00	.533333375795851822D+00	.000000000000000000D+00
-.133333336926956175D+01	.000000000000000000D+00	.240000005064119759D+01	.000000000000000000D+00	.952381028206878253D+00
.000000000000000000D+00	-.240000013256461003D+01	.000000000000000000D+00	.342857102275077293D+01	.000000000000000000D+00
-.533333340685817919D+00	.000000000000000000D+00	-.342857096423404976D+01	.000000000000000000D+00	.444444448620957494D+01
.000000000000000000D+00	-.952380957986810448D+00	.000000000000000000D+00	-.444444441598950713D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I4mr10 ARE AS FOLLOWS

.000000000000000000D+00	-.133333331075283858D+01	.000000000000000000D+00	-.533333328982473285D+00	.000000000000000000D+00
.133333334586287248D+01	.000000000000000000D+00	-.240000003893785295D+01	.000000000000000000D+00	-.952380981393499716D+00
.000000000000000000D+00	.240000019108133320D+01	.000000000000000000D+00	-.342857131533438878D+01	.000000000000000000D+00
.533333334834145692D+00	.000000000000000000D+00	.342857149088455830D+01	.000000000000000000D+00	-.444444441598950713D+01
.000000000000000000D+00	.952380925802612704D+00	.000000000000000000D+00	.444444454472629811D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I4mr11 ARE AS FOLLOWS

.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370051336902830D+03

THE VALUES OF THE INTEGRAL I5mr00 ARE AS FOLLOWS

.500000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.499999992549419403D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999988079710449D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999985098838806D+00

THE VALUES OF THE INTEGRAL I5mr01 ARE AS FOLLOWS

.000000000000000000D+00	.266666662150567716D+01	.000000000000000000D+00	.213333331592989314D+01	.000000000000000000D+00
-.666666672931436211D+00	.000000000000000000D+00	.360000005840677943D+01	.000000000000000000D+00	.238095245348374929D+01
.000000000000000000D+00	-.160000012738755546D+01	.000000000000000000D+00	.457142842044585171D+01	.000000000000000000D+00
-.133333333708536491D+00	.000000000000000000D+00	-.257142861816341872D+01	.000000000000000000D+00	.555555551998688391D+01
.000000000000000000D+00	-.380952370321045082D+00	.000000000000000000D+00	-.355555563578103849D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I5mr10 ARE AS FOLLOWS

.000000000000000000D+00	-.666666661228091606D+00	.000000000000000000D+00	-.133333334394896295 5D+00	.000000000000000000D+00
.266666673853912350D+01	.000000000000000000D+00	-.160000003376079839D+01	.000000000000000000D+00	-.380952411282751301D+00
.000000000000000000D+00	.360000019884691504D+01	.000000000000000000D+00	-.257142826706307970D+01	.000000000000000000D+00
.213333336274327168D+01	.000000000000000000D+00	.457142795231206635D+01	.000000000000000000D+00	-.355555558896765995D+01
.000000000000000000D+00	.238095239496702612D+01	.000000000000000000D+00	.555555551998688391D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I5mr11 ARE AS FOLLOWS

.493480227407896236D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132204667106612D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568363852633977D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370053175259906D+03

THE VALUES OF THE INTEGRAL I6mr00 ARE AS FOLLOWS

.000000000000000000D+00	-.212206589058041573D+00	.000000000000000000D+00	-.424413215368986130D-01	.000000000000000000D+00
.424413193017244339D+00	.000000000000000000D+00	-.254647914320230484D+00	.000000000000000000D+00	-.606304593384265900D-01
.000000000000000000D+00	.381971884518861771D+00	.000000000000000000D+00	-.272837013006210327D+00	.000000000000000000D+00
.169765274971723557D+00	.000000000000000000D+00	.363782677799463272D+00	.000000000000000000D+00	-.282942123711109161D+00
.000000000000000000D+00	.151576137170195580D+00	.000000000000000000D+00	.353677649050951004D+00	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I6mr01 ARE AS FOLLOWS

.157079635020158589D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.471238905060475766D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.628318540080634355D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.785398151694103675D+01

THE VALUES OF THE INTEGRAL I6mr10 ARE AS FOLLOWS

-.157079632679489662D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.471238891016462205D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.628318380915147331D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.785398139990759041D+01

THE VALUES OF THE INTEGRAL I6mr11 ARE AS FOLLOWS

.000000000000000000D+00	-.837758026769534909D+01	.000000000000000000D+00	-.670206427298370572D+01	.000000000000000000D+00
.418879024414909912D+01	.000000000000000000D+00	-.226194674728271297D+02	.000000000000000000D+00	-.149599654728222814D+02
.000000000000000000D+00	.150796459378304328D+02	.000000000000000000D+00	-.430846978262528409D+02	.000000000000000000D+00
.167551608662949719D+01	.000000000000000000D+00	.323135250242109993D+02	.000000000000000000D+00	-.698131696328040730D+02
.000000000000000000D+00	.598398563984106806D+01	.000000000000000000D+00	.558505373239974855D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I7mr00 ARE AS FOLLOWS

.000000000000000000D+00	-.212206589058041573D+00	.000000000000000000D+00	-.424413215368986130D-01	.000000000000000000D+00
.424413193017244339D+00	.000000000000000000D+00	-.254647914320230484D+00	.000000000000000000D+00	-.606304593384265900D-01
.000000000000000000D+00	.381971884518861771D+00	.000000000000000000D+00	-.272837013006210327D+00	.000000000000000000D+00
.169765274971723557D+00	.000000000000000000D+00	.363782677799463272D+00	.000000000000000000D+00	-.282942123711109161D+00
.000000000000000000D+00	.151576137170195580D+00	.000000000000000000D+00	.353677649050951004D+00	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I7mr01 ARE AS FOLLOWS

.157079635020158589D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.471238905060475766D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.628318540080634355D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.785398151694103675D+01

THE VALUES OF THE INTEGRAL I7mr10 ARE AS FOLLOWS

-.157079632679489662D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.471238891016462205D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.628318380915147331D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.785398139990759041D+01

THE VALUES OF THE INTEGRAL I7mr02 ARE AS FOLLOWS

.000000000000000000D+00	.837758034122963215D+01	.000000000000000000D+00	.670206486125797012D+01	.000000000000000000D+00
-.418879031768338217D+01	.000000000000000000D+00	.226194675831285542D+02	.000000000000000000D+00	.149599662081651119D+02
.000000000000000000D+00	-.150796455701590175D+02	.000000000000000000D+00	.430846941495386884D+02	.000000000000000000D+00
-.167551610501306795D+01	.000000000000000000D+00	-.323135200606468934D+02	.000000000000000000D+00	.698131707358183187D+02
.000000000000000000D+00	-.598398604206034645D+01	.000000000000000000D+00	-.558505357062432584D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I7mr20 ARE AS FOLLOWS

.000000000000000000D+00	.209439508530740804D+01	.000000000000000000D+00	.418879053828623132D+00	.000000000000000000D+00
-.167551612707335287D+02	.000000000000000000D+00	.100530967036126908D+02	.000000000000000000D+00	.239359459330641790D+01
.000000000000000000D+00	-.339292025328577894D+02	.000000000000000000D+00	.242351404591155122D+02	.000000000000000000D+00
-.268082576802090873D+02	.000000000000000000D+00	-.574462578855944772D+02	.000000000000000000D+00	.446804292709237240D+02
.000000000000000000D+00	-.373999127628771653D+02	.000000000000000000D+00	-.872664620410050912D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I7mr11 ARE AS FOLLOWS

.000000000000000000D+00	-.837758026769534909D+01	.000000000000000000D+00	-.670206427298370572D+01	.000000000000000000D+00
.418879024414909912D+01	.000000000000000000D+00	-.226194674728271297D+02	.000000000000000000D+00	-.149599654728222814D+02
.000000000000000000D+00	.150796459378304328D+02	.000000000000000000D+00	-.430846978262528409D+02	.000000000000000000D+00
.167551608662949719D+01	.000000000000000000D+00	.323135250242109993D+02	.000000000000000000D+00	-.698131696328040730D+02
.000000000000000000D+00	.598398583984106806D+01	.000000000000000000D+00	.558505373239974855D+02	.000000000000000000D+00

.000000000000000000D+00	.124025108569317388D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.418584728946648960D+03	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.992200617210476506D+03	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.193789223476504791D+04

THE VALUES OF THE INTEGRAL I7mr21 ARE AS FOLLOWS

-.155031385711646735D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.124025108569317388D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.418584741421446185D+03	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.992200868554539104D+03	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.193789226364189333D+04

THE VALUES OF THE INTEGRAL I7mr22 ARE AS FOLLOWS

.000000000000000000D+00	-.826834038062796682D+02	.000000000000000000D+00	-.661467288510580035D+02	.000000000000000000D+00
.165366813418593605D+03	.000000000000000000D+00	-.892980787234974627D+03	.000000000000000000D+00	-.590595793313017889D+03
.000000000000000000D+00	.133947122657498181D+04	.000000000000000000D+00	-.382705998298088382D+04	.000000000000000000D+00
.264586897986129207D+03	.000000000000000000D+00	.510274655688399772D+04	.000000000000000000D+00	-.110244540343717647D+05
.000000000000000000D+00	.147648937441940218D+04	.000000000000000000D+00	.137805673252384208D+05	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I8mr00 ARE AS FOLLOWS

.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999992549419403D+00

THE VALUES OF THE INTEGRAL I8mr01 ARE AS FOLLOWS

.000000000000000000D+00	.133333332245618321D+01	.000000000000000000D+00	.533333375795851822D+00	.000000000000000000D+00
-.133333336926956175D+01	.000000000000000000D+00	.240000005064119759D+01	.000000000000000000D+00	.952381028206878253D+00
.000000000000000000D+00	-.240000013256461003D+01	.000000000000000000D+00	.342857102275077293D+01	.000000000000000000D+00
-.533333340685817919D+00	.000000000000000000D+00	-.342857096423404976D+01	.000000000000000000D+00	.444444448620957494D+01
.000000000000000000D+00	-.952380957986810448D+00	.000000000000000000D+00	-.444444441598950713D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I8mr10 ARE AS FOLLOWS

.000000000000000000D+00	-.133333331075283858D+01	.000000000000000000D+00	-.533333328982473285D+00	.000000000000000000D+00
.133333334586287248D+01	.000000000000000000D+00	-.240000003893785295D+01	.000000000000000000D+00	-.952380981393499716D+00
.000000000000000000D+00	.240000019108133320D+01	.000000000000000000D+00	-.342857131533438878D+01	.000000000000000000D+00
.533333334834145602D+00	.000000000000000000D+00	.342857149088455830D+01	.000000000000000000D+00	-.444444441598950713D+01
.000000000000000000D+00	.952380925802612704D+00	.000000000000000000D+00	.444444454472629811D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I8mr11 ARE AS FOLLOWS

.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370051336902830D+03

THE VALUES OF THE INTEGRAL I9mr00 ARE AS FOLLOWS

.0000000000000000000D+00	-.2122065890580415730D+00	.0000000000000000000D+00	-.4244132153689861300D-01	.0000000000000000000D+00
.4244131930172443390D+00	.0000000000000000000D+00	-.2546479143202304840D+00	.0000000000000000000D+00	-.6063045933842659000D-01
.0000000000000000000D+00	.3819718645188617710D+00	.0000000000000000000D+00	-.2728370130062103270D+00	.0000000000000000000D+00
.1697652749717235570D+00	.0000000000000000000D+00	.3637826777994632720D+00	.0000000000000000000D+00	-.2829421237111091610D+00
.0000000000000000000D+00	.1515761371701955800D+00	.0000000000000000000D+00	.3536776490509510040D+00	.0000000000000000000D+00

THE VALUES OF THE INTEGRAL I9mr01 ARE AS FOLLOWS

.1570796350201585890D+01	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.3141592700403171770D+01	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.4712389050604757660D+01	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.6283185400806343550D+01	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.7853981516941036750D+01

THE VALUES OF THE INTEGRAL I9mr10 ARE AS FOLLOWS

-.1570796326794896620D+01	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	-.3141592700403171770D+01	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	-.4712388910164622050D+01	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	-.6283183809151473310D+01	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	-.7853981399907590410D+01

THE VALUES OF THE INTEGRAL I9mr11 ARE AS FOLLOWS

.0000000000000000000D+00	-.8377580267695349090D+01	.0000000000000000000D+00	-.6702064272983705720D+01	.0000000000000000000D+00
.4188790244149099120D+01	.0000000000000000000D+00	-.2261946747282712970D+02	.0000000000000000000D+00	-.1495996547282228140D+02
.0000000000000000000D+00	.1507964593783043280D+02	.0000000000000000000D+00	-.4308469782625284090D+02	.0000000000000000000D+00
.1675516086629497190D+01	.0000000000000000000D+00	.3231352502421099930D+02	.0000000000000000000D+00	-.6981316963280407300D+02
.0000000000000000000D+00	.5983985839841068060D+01	.0000000000000000000D+00	.5585053732399748550D+02	.0000000000000000000D+00

THE VALUES OF THE INTEGRAL I10mr00 ARE AS FOLLOWS

.5000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.5000000745058059700D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.4999999925494194030D+00	.0000000000000000000D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.4999998807907104490D+00	.0000000000000000000D+00
.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.0000000000000000000D+00	.4999999650988388060D+00

THE VALUES OF THE INTEGRAL I10mr01 ARE AS FOLLOWS

.0000000000000000000D+00	.2666666621505677160D+01	.0000000000000000000D+00	.2133333315929893140D+01	.0000000000000000000D+00
-.6666666729314362410D+00	.0000000000000000000D+00	.3600000058406779430D+01	.0000000000000000000D+00	.2380952453483749290D+01
.0000000000000000000D+00	-.1600000127387555460D+01	.0000000000000000000D+00	.4571428420445851710D+01	.0000000000000000000D+00
-.1333333337085361310D+00	.0000000000000000000D+00	-.2571428618163418720D+01	.0000000000000000000D+00	.5555555519986883910D+01
.0000000000000000000D+00	-.3809523703210450820D+00	.0000000000000000000D+00	-.3555555635781038490D+01	.0000000000000000000D+00

THE VALUES OF THE INTEGRAL I10mr10 ARE AS FOLLOWS

.0000000000000000000D+00	-.6666666612280916060D+00	.0000000000000000000D+00	-.1333333439489629550D+00	.0000000000000000000D+00
.2666666738539123500D+01	.0000000000000000000D+00	-.1600000337607983900D+01	.0000000000000000000D+00	-.3809524112827513010D+00
.0000000000000000000D+00	.3600000198846915040D+01	.0000000000000000000D+00	-.2571428267063079700D+01	.0000000000000000000D+00
.2133333627432716800D+01	.0000000000000000000D+00	.4571427952312066350D+01	.0000000000000000000D+00	-.3555555589676599500D+01
.0000000000000000000D+00	.2380952391967026120D+01	.0000000000000000000D+00	.5555555199868839100D+01	.0000000000000000000D+00

.000000000000000000D+00	-.263189446238505222D+02	.000000000000000000D+00	-.210551558838922285D+02	.000000000000000000D+00
.263189453168948121D+02	.000000000000000000D+00	-.142122305681493993D+03	.000000000000000000D+00	-.939962352547508736D+02
.000000000000000000D+00	.142122314691069767D+03	.000000000000000000D+00	-.406063710519276177D+03	.000000000000000000D+00
.210551561149069919D+02	.000000000000000000D+00	.406063731310604884D+03	.000000000000000000D+00	-.877298163368941275D+03
.000000000000000000D+00	.939962297681502424D+02	.000000000000000000D+00	.877298188780565251D+03	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I10mr22 ARE AS FOLLOWS

.487045455170012186D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.779272739884088036D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.394596812809100173D+04	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.124683606796627662D+05	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.304403400409329071D+05

THE VALUES OF THE INTEGRAL I11mr00 ARE AS FOLLOWS

.000000000000000000D+00	.424413174390792847D+00	.000000000000000000D+00	.169765271246433258D+00	.000000000000000000D+00
-.212206592783331871D+00	.000000000000000000D+00	.381971869617700577D+00	.000000000000000000D+00	.151576140895485878D+00
.000000000000000000D+00	-.254647929221391678D+00	.000000000000000000D+00	.363782715052366257D+00	.000000000000000000D+00
-.424413182772696018D-01	.000000000000000000D+00	-.272837050259113312D+00	.000000000000000000D+00	.353677649050951004D+00
.000000000000000000D+00	-.606304528191685677D-01	.000000000000000000D+00	-.2829421 2/436399460D+00	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I10mr02 ARE AS FOLLOWS

-.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.123370051336902830D+03

THE VALUES OF THE INTEGRAL I10mr20 ARE AS FOLLOWS

-.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.123370051336902830D+03

THE VALUES OF THE INTEGRAL I10mr11 ARE AS FOLLOWS

.493480227407896236D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132204667106612D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568363852633977D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370053175259906D+03

THE VALUES OF THE INTEGRAL I10mr12 ARE AS FOLLOWS

.000000000000000000D+00	.263189448548652856D+02	.000000000000000000D+00	.21055157/320103358D+02	.000000000000000000D+00
-.263189457789243392D+02	.000000000000000000D+00	.142122306374538284D+03	.000000000000000000D+00	.939962398750461419D+02
.000000000000000000D+00	-.142122311225848315D+03	.000000000000000000D+00	.406063675867061664D+03	.000000000000000000D+00
-.210551563459217553D+02	.000000000000000000D+00	-.406063668936618761D+03	.000000000000000000D+00	.877298177229827080D+03
.000000000000000000D+00	-.939962329446032394D+02	.000000000000000000D+00	-.877298163368941275D+03	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I10mr21 ARE AS FOLLOWS

THE VALUES OF THE INTEGRAL I11mr01 ARE AS FOLLOWS

-.157079632679489662D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.471238891016462205D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.628318380915147331D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.785398139990759041D+01

THE VALUES OF THE INTEGRAL I11mr10 ARE AS FOLLOWS

.157079635020158589D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.471238905060475766D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.628318540080634355D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.785398151694103675D+01

THE VALUES OF THE INTEGRAL I11mr11 ARE AS FOLLOWS

.000000000000000000D+00	.418879017061481607D+01	.000000000000000000D+00	.167551621531449253D+01	.000000000000000000D+00
-.837758063536676435D+01	.000000000000000000D+00	.150796450554190362D+02	.000000000000000000D+00	.598398648326604475D+01
.000000000000000000D+00	-.226194683552385263D+02	.000000000000000000D+00	.323135206121540163D+02	.000000000000000000D+00
-.670206442005227182D+01	.000000000000000000D+00	-.430846934141958579D+02	.000000000000000000D+00	.558505365886546550D+02
.000000000000000000D+00	-.149599651051508661D+02	.000000000000000000D+00	-.698131696328040730D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I12mr00 ARE AS FOLLOWS

.500000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.499999992549419403D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999880790710449D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999985098838806D+00

THE VALUES OF THE INTEGRAL I12mr01 ARE AS FOLLOWS

.000000000000000000D+00	.266666662150567716D+01	.000000000000000000D+00	.213333331592989314D+01	.000000000000000000D+00
-.666666672931436241D+00	.000000000000000000D+00	.360000005840677943D+01	.000000000000000000D+00	.238095245348374929D+01
.000000000000000000D+00	-.160000001273875546D+01	.000000000000000000D+00	.457142842044585171D+01	.000000000000000000D+00
-.133333333708536401D+00	.000000000000000000D+00	-.257142861816341872D+01	.000000000000000000D+00	.555555551998688391D+01
.000000000000000000D+00	-.380952370321045082D+00	.000000000000000000D+00	-.355555563578103849D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I12mr10 ARE AS FOLLOWS

.000000000000000000D+00	-.666666661228091606D+00	.000000000000000000D+00	-.133333343948962955D+00	.000000000000000000D+00
.266666673853912350D+01	.000000000000000000D+00	-.160000003376079839D+01	.000000000000000000D+00	-.380952411282751301D+00
.000000000000000000D+00	.360000019884691504D+01	.000000000000000000D+00	-.257142826706307970D+01	.000000000000000000D+00
.213333336274327168D+01	.000000000000000000D+00	.457142795231206635D+01	.000000000000000000D+00	-.355555558896765995D+01
.000000000000000000D+00	.238095239496702612D+01	.000000000000000000D+00	.555555551998688391D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I12mr11 ARE AS FOLLOWS

.493480227407896236D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132204667106612D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568363852633977D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370053175259906D+03

THE VALUES OF THE INTEGRAL I13mr00 ARE AS FOLLOWS

.000000000000000000D+00	-.212206589058041573D+00	.000000000000000000D+00	-.424413215368986130D-01	.000000000000000000D+00
.424413193017244339D+00	.000000000000000000D+00	-.254647914320230484D+00	.000000000000000000D+00	-.606304593384265900D-01
.000000000000000000D+00	.381971884516861771D+00	.000000000000000000D+00	-.272837013006210327D+00	.000000000000000000D+00
.169765274971723557D+00	.000000000000000000D+00	.363782677799463272D+00	.000000000000000000D+00	-.282942123711109161D+00
.000000000000000000D+00	.151576137170195580D+00	.000000000000000000D+00	.353677649050951004D+00	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I13mr01 ARE AS FOLLOWS

.157079635020158589D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.471238905060475766D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.628318540080634355D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.785398151694103675D+01

THE VALUES OF THE INTEGRAL I13mr10 ARE AS FOLLOWS

-.157079632679489662D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.471238891016462205D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.628318380915147331D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.785398139990759041D+01

THE VALUES OF THE INTEGRAL I13mr02 ARE AS FOLLOWS

.000000000000000000D+00	.837758034122963215D+01	.000000000000000000D+00	.670206486125797012D+01	.000000000000000000D+00
-.418879031768338217D+01	.000000000000000000D+00	.226194675831285542D+02	.000000000000000000D+00	.149599662081651119D+02
.000000000000000000D+00	-.150796455701590175D+02	.000000000000000000D+00	.430846941495386884D+02	.000000000000000000D+00
-.167551610501306795D+01	.000000000000000000D+00	-.323135200606468934D+02	.000000000000000000D+00	.698131707358183187D+02
.000000000000000000D+00	-.598398604206034645D+01	.000000000000000000D+00	-.558505357062432584D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I13mr20 ARE AS FOLLOWS

.000000000000000000D+00	.209439508530740804D+01	.000000000000000000D+00	.418879053828623132D+00	.000000000000000000D+00
-.167551612707335287D+02	.000000000000000000D+00	.100530967036126908D+02	.000000000000000000D+00	.239359459330641790D+01
.000000000000000000D+00	-.339292025328577894D+02	.000000000000000000D+00	.242351404591155122D+02	.000000000000000000D+00
-.268082576802090873D+02	.000000000000000000D+00	-.574462578855944772D+02	.000000000000000000D+00	.446804292709237240D+02
.000000000000000000D+00	-.373999127628771653D+02	.000000000000000000D+00	-.872664620410050912D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I13mr11 ARE AS FOLLOWS

.000000000000000000D+00	-.837758026769534909D+01	.000000000000000000D+00	-.670206427298370572D+01	.000000000000000000D+00
.418879024414909912D+01	.000000000000000000D+00	-.226194674728271297D+02	.000000000000000000D+00	-.149599654728222814D+02
.000000000000000000D+00	.150796459378304328D+02	.000000000000000000D+00	-.430846978262528409D+02	.000000000000000000D+00
.167551608662949719D+01	.000000000000000000D+00	.323135250242109993D+02	.000000000000000000D+00	-.698131696328040730D+02
.000000000000000000D+00	.598398583981068060D+01	.000000000000000000D+00	.558505373239974855D+02	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I13mr12 ARE AS FOLLOWS

.155031383401499101D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.124025108569317388D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.418591728946648960D+03	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.992200617210476506D+03	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.193789223476504791D+04

THE VALUES OF THE INTEGRAL I13mr21 ARE AS FOLLOWS

-.155031385711646735D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.124025108569317388D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.418584741421446185D+03	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.992200868554539104D+03	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.193789226364189333D+04

THE VALUES OF THE INTEGRAL I13mr22 ARE AS FOLLOWS

.000000000000000000D+00	-.826834038062796682D+02	.000000000000000000D+00	-.661467288510580035D+02	.000000000000000000D+00
.165366813418593605D+03	.000000000000000000D+00	-.892980787234974627D+03	.000000000000000000D+00	-.590595793313017889D+03
.000000000000000000D+00	.133947122657498181D+04	.000000000000000000D+00	-.382705998298088382D+04	.000000000000000000D+00
.264586897986129207D+03	.000000000000000000D+00	.510274655688399772D+04	.000000000000000000D+00	-.110244540343717647D+05
.000000000000000000D+00	.147648937441940218D+04	.000000000000000000D+00	.137805673252384208D+05	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I14mr00 ARE AS FOLLOWS

.500000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.500000007450580597D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.499999992549419403D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999988079071044D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.499999985098838806D+00

THE VALUES OF THE INTEGRAL I14mr01 ARE AS FOLLOWS

.000000000000000000D+00	.266666662150567716D+01	.000000000000000000D+00	.213333331592989314D+01	.000000000000000000D+00
-.666666672931436241D+00	.000000000000000000D+00	.360000005840677943D+01	.000000000000000000D+00	.238095245348374929D+01
.000000000000000000D+00	-.160000012738755546D+01	.000000000000000000D+00	.457142842044585171D+01	.000000000000000000D+00
-.133333333708536401D+00	.000000000000000000D+00	-.257142861816341872D+01	.000000000000000000D+00	.555555551998688391D+01
.000000000000000000D+00	-.380952370321945082D+00	.000000000000000000D+00	-.355555563578103849D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I14mr10 ARE AS FOLLOWS

.000000000000000000D+00	-.666666661228091606D+00	.000000000000000000D+00	-.133333343948962955D+00	.000000000000000000D+00
.266666673853912350D+01	.000000000000000000D+00	-.160000003376079839D+01	.000000000000000000D+00	-.380952411282751301D+00
.000000000000000000D+00	.360000019884691504D+01	.000000000000000000D+00	-.257142826706307970D+01	.000000000000000000D+00
.213333336274327168D+01	.000000000000000000D+00	.457142795231206635D+01	.000000000000000000D+00	-.355555558896765995D+01
.000000000000000000D+00	.238095239496702612D+01	.000000000000000000D+00	.555555551998688391D+01	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I14mr02 ARE AS FOLLOWS

-.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.123370051336902830D+03

THE VALUES OF THE INTEGRAL I14mr20 ARE AS FOLLOWS

-.493480220054467931D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.197392090963158494D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.444132191430935663D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.789568163839384081D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.123370051336902830D+03

THE VALUES OF THE INTEGRAL I14mr11 ARE AS FOLLOWS

.493480227407896236D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.197392090963158491D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.444132204667106612D+02	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.789568363852633977D+02	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.123370053175259906D+03

THE VALUES OF THE INTEGRAL I14mr12 ARE AS FOLLOWS

.000000000000000000D+00	.263189448548652856D+02	.000000000000000000D+00	.210551577320103358D+02	.000000000000000000D+00
-.263189457789243392D+02	.000000000000000000D+00	.142122306374538284D+03	.000000000000000000D+00	.939962398750461419D+02
.000000000000000000D+00	-.142122311225848315D+03	.000000000000000000D+00	.406063675867061664D+03	.000000000000000000D+00
-.210551563459217553D+02	.000000000000000000D+00	-.406063668936618761D+03	.000000000000000000D+00	.877298177229827080D+03
.000000000000000000D+00	-.939962329416032394D+02	.000000000000000000D+00	-.877298163368941275D+03	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I14mr21 ARE AS FOLLOWS

.000000000000000000D+00	-.263189446238505222D+02	.000000000000000000D+00	-.210551558838922285D+02	.000000000000000000D+00
.263189453168948124D+02	.000000000000000000D+00	-.142122305681493993D+03	.000000000000000000D+00	-.939962352547508736D+02
.000000000000000000D+00	.142122314691069767D+03	.000000000000000000D+00	-.406063710519276177D+03	.000000000000000000D+00
.217551561149069919D+02	.000000000000000000D+00	.406063731310604884D+03	.000000000000000000D+00	-.877298163368941275D+03
.000000000000000000D+00	.939962297681502424D+02	.000000000000000000D+00	.877298188780565251D+03	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I14mr22 ARE AS FOLLOWS

.487045455170012186D+02	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.779272739384088036D+03	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.394506812809100173D+04	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.124683606796627662D+05	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.304403400409329071D+05

THE VALUES OF THE INTEGRAL I15mr00 ARE AS FOLLOWS

.000000000000000000D+00	-.212206589058041573D+00	.000000000000000000D+00	-.424413215368986130D-01	.000000000000000000D+00
.424413193017241339D+00	.000000000000000000D+00	-.254647914320230484D+00	.000000000000000000D+00	-.606304593384265900D-01
.000000000000000000D+00	.381971884518861771D+00	.000000000000000000D+00	-.272837013006210327D+00	.000000000000000000D+00
.169765274971723557D+00	.000000000000000000D+00	.363782677799463272D+00	.000000000000000000D+00	-.282942123711109161D+00
.000000000000000000D+00	.151576137170195580D+00	.000000000000000000D+00	.353677649050951004D+00	.000000000000000000D+00

THE VALUES OF THE INTEGRAL I15mr01 ARE AS FOLLOWS

.157079635020158589D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.314159270640317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.471238905060475766D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.628318540806343355D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.785398151694103675D+01

THE VALUES OF THE INTEGRAL I15mr10 ARE AS FOLLOWS

-.157079632679489662D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	-.314159270040317177D+01	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	-.471238891016462205D+01	.000000000000000000D+00	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.628318380915147331D+01	.000000000000000000D+00
.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	.000000000000000000D+00	-.785398139990759041D+01

THE VALUES OF THE INTEGRAL I15mr11 ARE AS FOLLOWS

.000000000000000000D+00	-.837758026769534909D+01	.000000000000000000D+00	-.670206427298370572D+01	.000000000000000000D+00
.418879024414909912D+01	.000000000000000000D+00	-.226194674728271297D+02	.000000000000000000D+00	-.149599654728222814D+02
.000000000000000000D+00	.150796159378304328D+02	.000000000000000000D+00	-.430846978262528409D+02	.000000000000000000D+00
.167551608662949719D+01	.000000000000000000D+00	.323135250242109993D+02	.000000000000000000D+00	-.698131696328040730D+02
.000000000000000000D+00	.598398563981106806D+01	.000000000000000000D+00	.558505373239974855D+02	.000000000000000000D+00

Date Slip

This book is to be returned on the date last stamped.